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# SU(2|2) for Theories with Sixteen Supercharges at Weak and Strong Coupling

Abhishek Agarwal[1] and Donovan Young[2]

[1] *American Physical Society*
*1 Research Road*
*Ridge, NY 11961, USA*

[2] *Humboldt-Universität zu Berlin, Institut für Physik,*
*Newtonstraße 15, D-12489 Berlin, Germany*

[1] abhishek@ridge.aps.org, [2] dyoung@physik.hu-berlin.de

**Abstract**

We consider the dimensional reductions of $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory on $\mathbb{R} \times S^3$ to the three-dimensional theory on $\mathbb{R} \times S^2$, the orbifolded theory on $\mathbb{R} \times S^3/\mathbb{Z}_k$, and the plane-wave matrix model. With explicit emphasis on the three-dimensional theory, we demonstrate the realization of the $SU(2|3)$ algebra in a radial Hamiltonian framework. Using this structure we constrain the form of the spin chains, their S-matrices, and the corresponding one- and two-loop Hamiltonian of the three dimensional theory and find putative signs of integrability up to the two loop order. The string duals of these theories admit the IIA plane-wave geometry as their Penrose limit. Using known results for strings quantized on this background, we explicitly construct the strong-coupling dual extended $SU(2|2)$ algebra and discuss its implications for the gauge theories.

# Contents



# 1 Introduction and summary

Mass-deformed Lie superalgebras continue to play an important rôle in deepening our understanding of the gauge-gravity correspondence[1]. The algebras in question are generically of the form

$$\{Q^\dagger, Q\} = P + mR, \tag{1}$$

where $R$ is typically some combination of spacetime and R-symmetry rotation generators and $m$ denotes a basic "mass-gap" in the spectrum of observables constrained by the supersymmetry algebra. Perhaps the most dramatic manifestation of this algebraic structure is the integrability of the large-$N$ dilatation operator of $\mathcal{N} = 4$

[1] Because we will be dealing with string theory backgrounds which do not contain an $AdS$ subspace, and non-conformal dual field theories, we will avoid use of the more colloquial term "$AdS/CFT$" in this context.

supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM), where the "mass-gap" is the gap in the spectrum of conformal dimensions $\Delta$ of the operators of the theory, for which $\Delta \geq 2$. One of the basic building blocks on which much of the machinery that renders the four dimensional superconformal theory integrable rests is the closed $SU(2|3)$ sector. The matter fields in this sector comprise two Weyl fermions $\psi_\alpha$, $\alpha = 1, 2$, transforming under an $SU(2)$ of the Lorentz group $L$, and three complex scalars $X, Y, Z$, transforming under an $SU(3) \in SU(4) \sim SO(6)$ of the R-Symmetry group $R$. The relevant supercharges $S^\alpha_a$, $Q^b_\beta$, satisfy

$$\{S^\alpha_a, Q^b_\beta\} = \delta^b_a \delta^\alpha_\beta D + \delta^b_a L^\alpha_\beta + \delta^\alpha_\beta R^b_a, \tag{2}$$

where $a, b$ are $SU(2)$ indices corresponding to an $SU(2)$ contained in the R-symmetry group. This is nothing but a subalgebra of the full four dimensional superconformal algebra and $S, Q$ are a subset of the superconformal and supersymmetry generators respectively [1]. $D$ is the dilatation operator, which is realized as a quantum spin chain in the large-$N$ limit. The ferromagnetic ground state of the chain is spanned by the chiral primary operators $\mathrm{Tr}(Z^J)$. The excitations/magnons transform under the residual $SU(2|2)$ symmetry, which has profound consequences. For instance, the $S$-matrix of the spin chain, and its dispersion relation are both severely constrained to all orders in perturbation theory by this symmetry. The $SU(2|2)$ invariance also constrains the $S$-matrix to satisfy the Yang-Baxter equations [2, 3]. It is, of course, natural to ask if these compelling consequences of the underlying (mass-deformed) supersymmetry algebra have any repercussions for theories other than $\mathcal{N} = 4$ SYM.

Recent developments point out that the $SU(2|2)$ symmetry plays a greater rôle in the gauge-gravity correspondence than was previously realized. For example it appears in the studies of $\mathcal{N} = 6$ supersymmetric Chern-Simons (SCS) theories in two different contexts. The $\mathcal{N} = 6$ conformal models appear to posses an integrable dilatation operator in the large-$N$ limit [4, 5]. The dilatation operator is part of the center of a $SU(2|3)$ algebra just as in the case of the four dimensional superconformal theory mentioned above. A generalization of the $\mathcal{N} = 6$ SCS models can be obtained by adding appropriate mass terms to the matter fields. These mass-deformed models can be engineered to preserve $4 \leq \mathcal{N} \leq 8$ supersymmetry at the expense of conformal invariance. For such models $SU(2|2)$ algebras also arise as part of the spacetime supersymmetry, and they can be used to obtain all-loop results for the spacetime $S$-matrices of the massive theories [6]. These results clearly suggest that the search for other natural habitats for mass-deformed supersymmetry algebras and their consequences for the gauge/gravity duality are worthwhile endeavors.

A natural set of theories to investigate in this regard are the dimensional reductions of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. The $S^3$ can obviously be identified with $SU(2)$. Discarding the dependence of the degrees of freedom of the four dimensional theory on $\mathbb{Z}_k$, $U(1)$, and[2] $SU(2)$ produces 16 supercharge theories with massive spectra on $\mathbb{R} \times S^3/\mathbb{Z}_k$, $\mathbb{R} \times S^2$, and $\mathbb{R}$ respectively [7]. The theory on $\mathbb{R}$ is nothing but the plane-wave matrix model (PWMM) [8] while the other dimensional reductions result in gauge theories

[2]Both $Z_k$ and $U(1) \in SU(2) \sim S^3$.

with massive spectra[3]. Concrete proposals for the dual string theories corresponding to these dimensionally reduced models were also enunciated in [7]. Since the $SU(2|3)$ symmetry of the four dimensional gauge theory is preserved by these dimensional reductions it is imperative to try and uncover its consequences in the gauge/string dualities tying the lower dimensional non-conformal theories to string theories in non-$AdS$ type backgrounds. This line of investigation also has the advantage of being a valuable probe for the utility and robustness of the gauge-gravity conjecture for massive and non-conformal gauge theories and their dual string theory backgrounds. In much of the analysis that we perform, we concentrate on the three dimensional gauge theory and its string dual, while commenting on the generalizations of our results to the orbifold theory and the plane wave matrix model where we can do so.

Since much of our intuition about the use of mass-deformed superalgebras derives from studies of the dilatation operator of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$, it is instructive to recall how the conformal transformation mapping the theory to $\mathbb{R} \times S^3$ affects various elements of the superconformal algebra. In the flat background the superconformal algebra takes on the following heuristic form

$$\begin{aligned} \{Q, Q\} &= P, \\ \{S, S\} &= K, \\ \{S, Q\} &= D + R + L, \end{aligned} \tag{3}$$

where $K$ is the super-boost generator. Going from the flat space to $S^3$ amounts to radial quantization of the conformal theory. Under this quantization scheme, the generators map as follows [9]:

$$Q \to Q, \quad S \to Q^\dagger, \quad D \to H, \quad t \to r. \tag{4}$$

The relation between $Q^\dagger$ and $S$ is a consequence of the natural hermiticity properties endowed on the physical states upon the conformal transformation [9]. The last two relations in (4) simply imply that dilatations are mapped to radial scalings in the scheme of radial quantization, with the Hamiltonian assuming the rôle of $D$ [10]. These identifications enable us to recover a mass-deformed algebra (the analog of the last equation in (3)) even in the absence of conformal symmetries, as part of the supersymmetry algebra for the gauge theory on $\mathbb{R} \times S^3$. The SUSY algebra takes on the following generic form

$$\{Q^\dagger, Q\} = H + \mu R + \mu L, \tag{5}$$

where $\mu \sim$ (radius of $S^3$)$^{-1}$. It is important to note that the same basic algebraic structure is also valid for 16 supercharge theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$, $\mathbb{R} \times S^2$ and $\mathbb{R}$, even though, for these theories there is no natural sense in which they are equivalent to their massless counterparts. Recasting the details of the techniques applied in computing

[3]The mass for the scalars arises form the conformal coupling of the original four dimensional theory to the radius of $S^3$. The masses of the gluons are simply a consequence of the lack of a zero mode for vector fields on $S^2$ and $S^3$.

the spectrum of the dilatation operator of the four dimensional gauge theory and its string dual in the radial quantization scheme should then allow us to study and solve for the physical spectrum of these non-conformal gauge theories.

As mentioned above, this proposal has the potential of translating into a non-trivial test of the gauge-gravity conjecture in light of the existence of bona fide string duals for the non-conformal lower dimensional massive gauge theories. These are the Lin-Maldacena geometries [7], which we review in section 7. Quantizing the superstring in these backgrounds has only been successful in the plane-wave limit, yielding the same IIA plane-wave geometry for any of the Lin-Maldacena backgrounds[4]. Happily, the plane-wave limit is sufficient for uncovering the string-dual manifestation of the $SU(2|2)$ algebra we find in the gauge theories. The crucial element is the derivation of the central charges. For the full superstring on $AdS_5 \times S^5$, the authors of [11] showed insightfully that the central extension of the algebra follows from a relaxation of the level-matching condition, the natural dual of the length-changing action in the gauge theory. We show that the very same mechanism is at play for the plane-wave limit of the Lin-Maldacena geometries, and this allows us a full exhibition of the $SU(2|2)$ algebra in that limit. Under the assumption that the algebra persists beyond the plane-wave limit, we are able to discuss, on a qualitative level, the finite-size corrections, worldsheet scattering matrix, spinning string and giant magnon solutions associated with these string sigma models.

This results in this paper are organized as follows. In section 2 we explicitly construct the $SU(2|3)$ algebra in SYM on $\mathbb{R} \times S^3$ and on $\mathbb{R} \times S^2$. In sections 3 and 4 we obtain the dispersion relation for SYM on $\mathbb{R} \times S^2$, and constrain the form of the one and two-loop effective Hamiltonian. We also present evidence of integrability at the two loop level for the three dimensional theory. In section 5 we show the natural generalization of the $SU(2|2)$ $S$-matrix from SYM on $\mathbb{R} \times S^3$ to SYM on $\mathbb{R} \times S^2$. Our results concerning the universal form of the dispersion relation and S-matrix are expected to remain valid to all loop orders. However, as far as the explicit forms of the effective Hamiltonians and statements about integrability are concerned, the present gauge theoretic analysis is restricted to the two-loop order. In section 6 we discuss the extension of our results to the PWMM and SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$. In section 7 we continue the analysis to the leading order at strong coupling using the dual string theory. We discuss the Lin-Maldacena geometries and review the quantization of the string on the IIA plane-wave. We then derive the $SU(2|2)$ algebra in the plane-wave setting and discuss implications and further directions. Finally, we end with a discussion in section 8.

## 2 $SU(2|2)$ in SYM on $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2$

In this section, we isolate the closed $SU(2|3)$ sector in the sixteen supercharge Yang-Mills theories on $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2$. We start with the four dimensional theory in

[4]There is a caveat here concerning the vacuum of the gauge theory under consideration. The plane-wave geometry is only valid for "well-spaced" vacua. The meaning of "well spaced" is described in section 7.

radial quantization and present the details of the emergence of the algebraic structure (5) in a Hamiltonian picture. The analysis of the four dimensional model also allows us to calibrate and verify our results against known results for the dilatation operator for the gauge theory in flat background geometries. We find it convenient to use the conventions used in [12], and use the action

$$
\begin{aligned}
S = \frac{1}{g^2}\,\mathrm{Tr}\int_{\mathbb{R}\times S^3}\Bigg[&-\frac{1}{4}F_{ab}^2 - \frac{1}{2}D_a X_m D^a X^m - \frac{1}{2}\left(\frac{\mu}{2}\right)^2 X_m X^m - \frac{i}{2}\bar{\lambda}\Gamma^a D_a \lambda \\
&+\frac{1}{4}[X_m, X_n]^2 - \frac{1}{2}\bar{\lambda}\Gamma^m[X_m,\lambda]\Bigg],
\end{aligned}
\tag{6}
$$

where $m, n = 1, \ldots, 6$ are $SO(6)$ indices and $a, b = 0, \ldots, 3$ are spacetime indices. It is understood that we have normalized the radius of $S^3$ to be $2/\mu$, such that the volume is given by $2\pi^2(2/\mu)^3$. In other words,

$$
\int_{\mathbb{R}\times S^3} \equiv \left(\frac{1}{8}\right)\left(\frac{2}{\mu}\right)^3 \int dt \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi. \tag{7}
$$

We can always adjust the radius to be any other number by correspondingly scaling the coefficient of the "mass-term" for the scalars. $\Gamma^M = (\gamma^a \otimes I_8, \Gamma^m)$ are the ten-dimensional gamma matrices, while $\gamma^a$ are the four dimensional ones. The ten-dimensional spinor $\lambda$ is decomposed in terms of four-dimensional spinors as

$$
\lambda = \begin{pmatrix} \lambda_+^A \\ \lambda_{-A} \end{pmatrix}, \quad \text{with} \quad \lambda_+^A = \begin{pmatrix} \Psi_\alpha \\ 0 \end{pmatrix}, \tag{8}
$$

where the $SU(4)$ index $A = 1, \ldots, 4$ and $\lambda_+^A$ is a positive chirality, four-dimensional spinor, so that $\Psi_\alpha$ carries an $SU(2)$ index $\alpha = 1, 2$, i.e. it is a complex 2-spinor. The negative chirality counter-part is given by $\lambda_{-A} = C_4(\bar{\lambda}_{+A})^T$, where $C_4$ is the four-dimensional charge conjugation operator, see [12] for details. The covariant derivatives $D_a = \nabla_a - i[A_a, \;]$, where

$$
\begin{aligned}
\nabla_a A_b &= e_a^\mu(\partial_\mu A_b + \omega_{\mu b}{}^c A_c), \\
\nabla_a X^m &= e_a^\mu \partial_\mu X^m, \\
\nabla_a \lambda &= e_a^\mu(\partial_\mu \lambda + \frac{1}{4}\omega_\mu{}^{bc}\Gamma_{bc}\lambda),
\end{aligned}
\tag{9}
$$

where $\mu = t, \theta, \phi, \psi$ is a curved-space index and $a, b, c$ are tangent-space indices. The non-vanishing components of the vierbeins and spin connections are

$$
\begin{aligned}
&e_\theta^1 = 1/\mu, \quad e_\phi^2 = \frac{\sin\theta}{\mu}, \quad e_\phi^3 = \frac{\cos\theta}{\mu}, \quad e_\psi^3 = 1/\mu, \\
&e_1^\theta = \mu, \quad e_2^\phi = \frac{\mu}{\sin\theta}, \quad e_2^\psi = -\mu\frac{\cos\theta}{\sin\theta}, \quad e_3^\psi = \mu, \\
&\omega_{12} = -\omega_{21} = -\frac{1}{2}(\cos\theta d\phi - d\psi), \quad \omega_{23} = -\omega_{32} = -\frac{1}{2}d\theta, \\
&\omega_{31} = -\omega_{13} = -\frac{1}{2}\sin\theta d\phi.
\end{aligned}
\tag{10}
$$

The supercharges $Q$, like the fermionic fields, are decomposed as

$$Q = \left( \begin{array}{c} Q_+^A \\ Q_{-A} \end{array} \right), \tag{11}$$

with $Q_{-A} = C_4(\bar{Q}_{+A})^T$. Although the $SO(6)$ basis, in which the scalar fields and gamma matrices are represented as $X^m$ and $\Gamma^m$, provides a compact expression for the action (6), we will need to express the supercharges of the theory in an $SU(4)$ basis where we have instead $X^{AB}$, $A, B = 1, \ldots, 4$, and similarly for the $\Gamma^m$. The dictionary between the two bases is given in appendix B. We adopt a Hamiltonian formalism with $A_0 = 0$. In the canonical formalism, the explicit expressions for the supercharges are

$$\begin{aligned} Q^*_{+A} = \frac{1}{g^2} \operatorname{Tr} \int_{S^3} \Bigg[ & g^2 \lambda^*_{+A} \gamma^i E_i + \frac{1}{2} \lambda^*_{+A} \gamma^{ij} \gamma^0 F_{ij} - 2g^2 \Pi_{AB} \lambda_-^{*B} \gamma^5 - 2(D_i X_{AC}) \lambda_-^{*C} \gamma^i \gamma^5 \gamma^0 \\ & \pm 2i \left( \frac{\mu}{2} \right) X_{AC} \lambda_-^{*C} \gamma^5 - 2i [X_{AL}, X^{LP}] \lambda^*_{+P} \gamma^0 \Bigg], \end{aligned} \tag{12}$$

$$\begin{aligned} Q_+^A = \frac{1}{g^2} \operatorname{Tr} \int_{S^3} \Bigg[ & g^2 \gamma^i E_i \lambda_+^A + \frac{1}{2} \gamma^0 \gamma^{ij} \lambda_+^A F_{ij} - 2g^2 \Pi^{AB} \gamma^5 \lambda_{-B} + 2(D_i X^{AC}) \gamma^0 \gamma^5 \gamma^i \lambda_{-C} \\ & \mp 2i \left( \frac{\mu}{2} \right) X^{AM} \gamma^5 \lambda_{-M} + 2i [X^{AL}, X_{LP}] \gamma^0 \lambda_+^P \Bigg], \end{aligned} \tag{13}$$

where $E_i = g^{-2} \dot{A}_i$, $\Pi^{AB} = g^{-2} \dot{X}^{AB}$, and where we have introduced the spatial index $i, j = 1, \ldots, 3$ so that $a = (0, i)$.

The supersymmetry variation of a generic field $\mathcal{W} \to \delta_\epsilon \mathcal{W} = [\bar{Q}_{+A} \epsilon_+^A + \bar{Q}_-^A \epsilon_{-A}, \mathcal{W}]$, where the spinor $\epsilon$ satisfies the conformal Killing equation

$$\nabla_\mu \epsilon_+^A = \pm \frac{i\mu}{4} \gamma_\mu \gamma^0 \epsilon_+^A. \tag{14}$$

The two signs on the *r.h.s.* of the Killing equation result in the signs in front of the "mass-terms" (the terms linear in $X_{AB}$ which do not involve derivatives) in the expressions for the supercharges presented above. In what is to follow, we shall take the upper sign in the Killing equation, which would correspond to the lower sign in front of the "mass-terms" in the expression for the supercharges.

The canonical commutation relations following from the action are given by

$$\begin{aligned} & [X_{AB}(x), \dot{X}^{CD}(y)] = ig^2 \frac{1}{2} \delta^3(x-y) \left( \delta_A^C \delta_B^D - \delta_B^C \delta_A^D \right), \\ & [A_i(x), \dot{A}_j(y)] = ig^2 \delta^3(x-y)\, \delta_{ij}, \\ & \{\lambda_{+A}(x), \lambda_+^{\dagger B}(y)\} = g^2 \delta^3(x-y)\, \delta_A^B, \\ & \{\lambda_-^A(x), \lambda_{B-}^\dagger(y)\} = g^2 \delta^3(x-y)\, \delta_B^A. \end{aligned} \tag{15}$$

It is useful to extract the action of the supercharges on single-particle states formed by the scalar and fermionic partons of the theory. For this purpose it is useful to introduce the oscillators

$$\begin{aligned}
&\alpha^{AB} = \sqrt{\frac{\mu}{2g^2}}X^{AB} + i\frac{1}{\sqrt{\frac{\mu}{2g^2}}}\Pi^{AB}, \quad \alpha^\dagger_{AB} = \sqrt{\frac{\mu}{2g^2}}X_{AB} - i\frac{1}{\sqrt{\frac{\mu}{2g^2}}}\Pi_{AB}, \\
&[\alpha^{AB}, \alpha^\dagger_{CD}] = \delta^3(x-y)\left(\delta^A_C\delta^B_D - \delta^A_D\delta^B_C\right).
\end{aligned} \tag{16}$$

Notice, that these oscillators differ from the oscillator variables usually employed in the canonical quantization of massive scalar fields. We have *not* Fourier decomposed any of the spacetime coordinates, and the oscillator variables depend on the three $S^3$ coordinates as well as on time. The vacuum of the field theory is taken to be annihilated by $\alpha_{AB}$ and $\lambda_+$. On the single particle states built out of the scalar and fermionic fields

$$\begin{aligned}
&[Q^A_+, \alpha^\dagger_{MN}]|0\rangle = 2\sqrt{\frac{\mu}{2g^2}}\left(+\delta^A_N\begin{pmatrix} 0 \\ \sigma^2\Psi^*_M \end{pmatrix} - \delta^A_M\begin{pmatrix} 0 \\ \sigma^2\Psi^*_N \end{pmatrix}\right)|0\rangle, \\
&[Q^*_{+A}, \alpha^\dagger_{MN}]|0\rangle = 0, \\
&\{Q^A_{+\alpha}, \lambda^*_{B\beta}\}|0\rangle = \left[\left(g^2E_i\gamma^i_{\alpha\beta} + \frac{1}{2}F_{ij}(\gamma^0\gamma^{ij})_{\alpha\beta}\right)\delta^A_B + 2i[X^{AL}, X_{LB}]\gamma^0_{\alpha\beta}\right]|0\rangle, \\
&\{Q^*_{+A\alpha}, \lambda^*_{+B\beta}\}|0\rangle = -2i\sqrt{\frac{\mu g^2}{2}}(\gamma^5C_4\gamma^0)_{\alpha\beta}\alpha^\dagger_{AB}|0\rangle.
\end{aligned} \tag{17}$$

The above relations are true only modulo the equations of motion and spatial translations, as they would be for any supersymmetric Yang-Mills theory.

To proceed further, it is instructive to fix our conventions such that

$$\gamma^0 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & +\sigma^2 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{18}$$

The two bosonic and two fermionic states transforming under $SU(2)_R$ and $SU(2)_L$ can be taken to be

$$|\phi_a\rangle = \alpha^\dagger_{4a(=1,2)}|0\rangle, \quad |\psi_\alpha\rangle = \Psi^*_{4\alpha}|0\rangle. \tag{19}$$

A note about the positions of the fermionic $SU(2)$ indices is in order. $\Psi^M = \Psi^M_\alpha$ and $\Psi^*_M = \Psi^{*\alpha}_M$ are the natural positions of the $SU(2)$ index "$\alpha$" on the two component complex spinor $\Psi$. However in creating the state $|\psi_\alpha\rangle$ the index is lowered using $\Psi^*_\alpha = \epsilon_{\alpha\beta}\Psi^{*\alpha}$. It is also understood that $\epsilon_{12} = -\epsilon^{12} = 1$. After restricting $A, B = 1, 2$ on $Q, Q^*$ and renaming the restricted supercharges $q^a_\alpha, q^{*\alpha}_a$, we obtain the fundamental representation of $SU(2|2)$ which can be expressed manifestly as

$$\begin{aligned}
&q^a_\alpha|\phi_b\rangle = -2i\sqrt{\frac{\mu}{2g^2}}\delta^a_b|\psi_\alpha\rangle, \\
&q^{*\alpha}_a|\phi_b\rangle = 0, \\
&q^a_\alpha|\psi_\beta\rangle = 2\epsilon_{\alpha\beta}\epsilon^{ab}|[\phi_b, Z]\rangle, \\
&q^{*\alpha}_a|\psi_\beta\rangle = +2i\sqrt{\frac{\mu g^2}{2}}\delta^\alpha_\beta|\phi_a\rangle.
\end{aligned} \tag{20}$$

The canonical anti-commutation relation between the supercharges is given by

$$\{q^a_\alpha, q^{*\beta}_b\} = 2\delta^a_b\delta^\beta_\alpha H + 4\left(\frac{\mu}{2}\right)\delta^\beta_\alpha \mathcal{R}^a_b + 4\left(\frac{\mu}{2}\right)\delta^a_b \mathcal{L}^\beta_\alpha, \tag{21}$$

where

$$\begin{aligned}\mathcal{R}^a_b &= \sum_{C=1}^{4} \alpha^\dagger_{bC}\alpha^{aC} - \delta^a_b \frac{1}{4}\sum_{M,N=1}^{4} \alpha^\dagger_{MN}\alpha^{MN},\\ \mathcal{L}^\alpha_\beta &= \Psi^{*\alpha}_4 \Psi^4_\beta.\end{aligned} \tag{22}$$

This completes our discussion of a concrete realization of (5) in the context of the Hamiltonian formulation of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. As expected, the rôle of the conformal dimensions is assumed by the masses of the collective excitations of the gauge theory in the curved background, with the scale of the masses being set by $\mu$.

## 2.1 Dimensional reductions

We shall now work out the dimensional reductions of the Hamiltonian and the supercharges to $\mathbb{R} \times S^2$. To carry out the dimensional reduction to the three dimensional spacetime, we need to assume that the scalar and fermionic fields do not depend on the $U(1)$ coordinate $\psi$. As for the gauge field, one simply replaces the third component of the one-form on the tangent space by a scalar; namely $A_3 = \Phi$. More concretely

$$\begin{aligned}A &= A_a e^a_\mu dx^\mu = e^\mu_a A_\mu e^a_\nu dx^\nu\\ &= A_t dt + A_\theta d\theta + A_\phi d\phi + \frac{\Phi}{\mu}(\cos\theta d\phi + d\psi).\end{aligned} \tag{23}$$

Using this decomposition in (6) and dropping the $\psi$ dependence of all the fields, we get

$$\begin{aligned}S = \frac{1}{g^2_{S^2}} \operatorname{Tr} \int_{\mathbb{R}\times S^2} \Bigg[ &- \frac{1}{4}F^2_{ab} - \frac{1}{2}D_a\Phi D^a\Phi - \frac{\mu^2}{2}\Phi^2 + \mu F_{12}\Phi\\ &- \frac{1}{2}D_a X_m D^a X^m - \frac{1}{2}\left(\frac{\mu}{2}\right)^2 X_m X^m - \frac{i}{2}\bar\lambda\Gamma^a D_a\lambda\\ &+ \frac{i\mu}{8}\bar\lambda\Gamma^{123}\lambda - \frac{1}{2}\bar\lambda\Gamma^3[\Phi,\lambda]\\ &+ \frac{1}{2}[\Phi, X_m]^2 + \frac{1}{4}[X_m, X_n]^2 - \frac{1}{2}\bar\lambda\Gamma^m[X_m,\lambda]\Bigg].\end{aligned} \tag{24}$$

The radius of $S^2$ is $1/\mu$, with the non-vanishing dreibeins and spin connections given by the standard formulae

$$\begin{aligned}b^1_\theta &= \frac{1}{b^\theta_1} = \frac{1}{\mu},\ \ b^2_\phi = \frac{1}{b^\phi_2} = \frac{\sin\theta}{\mu},\\ \omega_{12} &= -\omega_{21} = -\cos\theta d\phi.\end{aligned} \tag{25}$$

The three dimensional coupling $g_{S^2}^2$ is related to the four dimensional one as

$$g_{S^2}^2 = \frac{g^2\mu}{4\pi}. \tag{26}$$

The measure of integration

$$\int_{\mathbb{R}\times S^2} \equiv \frac{1}{\mu^2}\int dt \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi, \tag{27}$$

is defined to yield a volume of $4\pi/\mu^2$. This dimensional reduction yields the following expression for the supercharge for the three dimensional theory

$$\begin{aligned} Q_+^A = \frac{1}{g_{S^2}^2} \operatorname{Tr} \int_{S^2} \Bigg[ & g_{S^2}^2 \gamma^i E_i \lambda_+^A + \frac{1}{2}\gamma^0\gamma^{ij}\lambda_+^A F_{ij} - 2g_{S^2}^2 \Pi^{AB}\gamma^5\lambda_{-B} \\ & + 2(D_i X^{AC})\gamma^0\gamma^5\gamma^i\lambda_{-C} + 2i\left(\frac{\mu}{2}\right) X^{AM}\gamma^5\lambda_{-M} \\ & + 2i[X^{AL}, X_{LP}]\gamma^0\lambda_+^P + g_{S^2}^2 \Pi_\Phi \gamma^3 \lambda_+^A \\ & + \gamma^0\gamma^{i3} D_i \Phi \lambda_+^A - i\mu\Phi\gamma^3\lambda_+^A - 2i[\Phi, X^{AC}]\gamma^0\gamma^5\gamma^3\lambda_{-C} \Bigg], \end{aligned} \tag{28}$$

The last line contains all the terms involving the new (seventh) scalar field obtained from the dimensional reduction of the four dimensional vector potential.

Comparing the expression for the dimensionally reduced supercharge with (13), it is easy to see that it admits a restriction to an $SU(2|3)$ sector, just like (20). However, we have not yet shown that the supercharge presented above is indeed a symmetry of the Hamiltonian obtained by the dimensional reduction to $\mathbb{R}\times S^2$. To do that, we need to reproduce the supercharge as the time component of a supercurrent, which we shall now proceed to do. To this end, it is instructive to recall that for $\mathcal{N}=1$ SYM in 10-d,

$$\mathcal{L} = -\frac{1}{2}F_{MN}F^{MN} + i\bar{\Psi}\Gamma^M D_M \Psi, \tag{29}$$

the SUSY variations of the fields

$$\delta A_N = -2i\bar{\Psi}\Gamma_N\epsilon, \qquad \delta\Psi = F_{PQ}\Gamma^{PQ}\epsilon, \tag{30}$$

produce the supercurrent

$$j^M = 2i\bar{\Psi}\Gamma^M\Gamma^{PQ}F_{PQ}\epsilon. \tag{31}$$

The supercharge is then given by

$$Q = \int_{\text{space}} j^0 = 2i\int_{\text{space}} \bar{\Psi}\Gamma^0\Gamma^{PQ}F_{PQ}\epsilon. \tag{32}$$

Keeping the ten dimensional theory in mind, one can write the Lagrangian for the $\mathbb{R}\times S^2$ theory in the following form [13]

$$\mathcal{L} = -\frac{1}{2}F_{MN}F^{MN} + i\bar{\Psi}\Gamma^M\nabla_M\Psi - i\frac{\mu}{4}\bar{\Psi}\Gamma^{123}\Psi + 2\mu\Phi F_{12} - \frac{\mu^2}{4}\phi_{\bar{m}}^2 - \mu^2\Phi^2, \tag{33}$$

where the 10-d gauge field is understood as $A_M = (A_\mu, \phi_m) = (A_\mu, \Phi, \phi_{\bar{m}})$ with $\mu = 0,1,2$ and $m = 3,\ldots,9$, $\bar{m} = 4,\ldots,9$. The SUSY variations of the fields are expressed as

$$\delta A_M = -2i\bar{\Psi}\Gamma_M\epsilon, \qquad \delta\Psi = F_{MN}\Gamma^{MN}\epsilon - \mu\Gamma^m\Gamma^{123}\phi_m\epsilon - \mu\Gamma^3\Gamma^{123}\Phi\epsilon. \tag{34}$$

Since the kinetic part of the action is not different from $\mathcal{N}=1$ SYM in 10-d the $\mu$-independent part of the supercurrent that does not involve total derivatives will be the same as in the ten dimensional theory. Extra total derivative (surface terms) are generated from the $\mu$-dependent piece of the variation of the fermion kinetic term, whose contribution to $\delta\mathcal{L}$ is

$$\nabla_M\left(i\bar{\Psi}\Gamma^M\left(\mu\Gamma^m\Gamma^{123}\phi_m + \mu\Gamma^3\Gamma^{123}\Phi\right)\epsilon\right). \tag{35}$$

Of course the very same term is generated by the new $\mu$-dependent piece of $\frac{\delta\mathcal{L}}{\delta(\partial_M\Psi)}\delta\Psi$. These add in the expression for the supercurrent, giving us

$$Q = \int_{S^2}\left(2i\bar{\Psi}\Gamma^0\Gamma^{PQ}F_{PQ}\epsilon - 2i\mu\bar{\Psi}\Gamma^0\Gamma^m\Gamma^{123}\phi_m\epsilon - 2i\mu\bar{\Psi}\Gamma^0\Gamma^3\Gamma^{123}\Phi\epsilon\right), \tag{36}$$

which is thus the supercharge for the $\mathbb{R}\times S^2$ theory, albeit in a rather compact notation. Expressing the $SO(6)$ fields in terms of $SU(4)$ ones using the dictionary in appendix B, we find

$$\begin{aligned} Q^A_+ =4i\int_{S^2}\Bigg(&\frac{1}{2}\gamma^{\mu\nu}\gamma^0F_{\mu\nu}\,\lambda^A_+ - 2\gamma^\mu\gamma^0D_\mu X^{AB}\lambda_{-B} + \gamma^\mu\gamma^3\gamma^0D_\mu\Phi\,\lambda^A_+ \\ &+ 2ig\gamma^3\gamma^0[\Phi, X^{AB}]\lambda_{-B} + 2ig\gamma^0[X^{AC}, X_{CB}]\lambda^B_+ - i\mu X^{AB}\lambda_{-B} - i\mu\gamma^3\Phi\,\lambda^A_+\Bigg). \end{aligned} \tag{37}$$

This agrees with (28) up to the overall $4i$ outside, which can be easily absorbed in a redefinition of the charge.

The construction clearly shows that we can restrict the three dimensional theory consistently to an $SU(2|3)$ sector. Furthermore, the reduced supercharges act on the $SU(2|3)$ states exactly as in (20), with $g$ replaced by $g_{S^2}$. The three dimensional supercharges constructed above satisfy the same massive algebra (21) as the four dimensional theory allowing us to constrain its quantum spectrum on algebraic grounds as discussed below.

# 3 Dispersion relations and the extended $SU(2|2)$ algebra

In this section we shall focus on constraining the spectrum of the four and three dimensional sixteen supercharge theories in the scheme of radial quantization. We

shall put special emphasis on the rôle played by the scale introduced by the radius of the sphere $1/\mu$. Following that we shall extend the formalism to incorporate the three dimensional $\mathcal{N}=8$ theory. Following [2, 3] we write the $SU(2|2)$ algebra (20) abstractly as

$$\begin{aligned} q_\alpha^a|\phi_b\rangle &= a\,\delta_b^a|\psi_\alpha\rangle, \\ q_a^{*\alpha}|\phi_b\rangle &= c\,\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\beta\rangle, \\ q_\alpha^a|\psi_\beta\rangle &= b\,\epsilon_{\alpha\beta}\epsilon^{ab}|\phi_b\rangle, \\ q_a^{*\alpha}|\psi_\beta\rangle &= d\,\delta_\beta^\alpha|\phi_a\rangle. \end{aligned} \tag{38}$$

In the fundamental representation, which corresponds to the tree-level field theory, one has $a=-2i\sqrt{\mu/2g^2}$, $b=c=0$, and $d=+2i\sqrt{\mu g^2/2}$. To proceed beyond the classical theory, one needs to augment this algebra by two new generators $P, K$ defined by [2, 3]

$$\begin{aligned} \{q_\alpha^a, q_\beta^b\} &= \epsilon^{ab}\epsilon_{\alpha\beta}P = \epsilon^{ab}\epsilon_{\alpha\beta}ab \\ \{q_a^{*\alpha}, q_b^{*\beta}\} &= \epsilon^{\alpha\beta}\epsilon_{ab}K = \epsilon^{\alpha\beta}\epsilon_{ab}cd \end{aligned} \tag{39}$$

where $P$ and $K$ annihilate physical states. Notice that unlike the case of the gauge theory on $\mathbb{R}^4$, $P$ and $K$ are not independent, as $P=K^*$. This is yet another artifact of the effect of the conformal transformations that map the theory from flat spacetime to the sphere. The conformal transformation maps the superconformal generators to the conjugates of the supercharges, and the relation between $P$ and $K$ is another reflection of the same map.

Closure of the algebra on $H$ and the rotation generators yields

$$H = \frac{1}{4}(ad+bc) \quad \text{and} \quad (ad-bc) = 2\mu. \tag{40}$$

The second, level-shortening condition, is easily checked to be satisfied at the classical level, using the values of $a, b, c$, and $d$ in (20). These relations also yield the dispersion relation for the magnons

$$H = \frac{1}{2}\sqrt{\mu^2 + PK}. \tag{41}$$

All the statements made above hold both for the four and three dimensional gauge theories. However, for the specific case of the three dimensional theory, we would like to emphasize that its Hamiltonian involves the "perfect square" term $(F_{12}-\mu\Phi)^2$, whose minima generate the moduli space of the vacua of this theory. Our results for SYM on $\mathbb{R}\times S^2$ only apply to the trivial vacuum $\Phi=0$. From the viewpoint of the gauge-gravity duality, the string dual (64) for the theory proposed in [7] and studied later in the paper applies only to this vacuum, making it particularly interesting[5].

[5]Other vacua of the gauge theory correspond to monopole backgrounds [12]. It would doubtless be interesting to understand the quantum spectra of the theory around these non-trivial vacua as well.

To proceed further, it is important to parameterize the algebra as

$$\begin{aligned}
a &= \sqrt{\mu h(\lambda)}\eta \\
b &= \sqrt{\mu h(\lambda)}\frac{\zeta}{\eta}(1 - x^+/x^-) \\
c &= \sqrt{\mu h(\lambda)}\frac{i\eta}{\zeta x^+} \\
d &= \sqrt{\mu h(\lambda)}\frac{x^+}{i\eta}(1 - x^-/x^+)
\end{aligned} \tag{42}$$

The above parameterization, where $h$ is an arbitrary function of the dimensionless 't Hooft coupling of the gauge theory is specific to the case of the $\mathcal{N} = 4$ theory on $\mathbb{R} \times S^3$. That is so because the length dimensions of the four parameters are all equal to $-1/2$ only in the four dimensional theory. It is easily seen that for the $\mathbb{R} \times S^2$ case, $a$ and $c$ are dimensionless, while $b$ and $d$ have length dimension $-1$. The appropriate parameterization in that case is

$$\begin{aligned}
a &= \sqrt{h(\lambda)}\eta, \\
b &= \mu\sqrt{h(\lambda)}\frac{\zeta}{\eta}(1 - x^+/x^-), \\
c &= \sqrt{h(\lambda)}\frac{i\eta}{\zeta x^+}, \\
d &= \mu\sqrt{h(\lambda)}\frac{x^+}{i\eta}(1 - x^-/x^+).
\end{aligned} \tag{43}$$

However, in both cases the shortening condition implies

$$\frac{2i}{h(\lambda)} = x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-}. \tag{44}$$

Also, writing $x^+/x^- = e^{ik}$ allows us to write

$$\begin{aligned}
P &= ab = \mu h(\lambda)(1 - e^{ik}), \\
K &= cd = \mu h(\lambda)(1 - e^{-ik}),
\end{aligned} \tag{45}$$

for both theories in question. Thus the dispersion relation for both the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories can naturally be expressed as

$$H = \frac{\mu}{2}\sqrt{1 + 4h^2(\lambda)\sin^2(k/2)}. \tag{46}$$

Obviously, in the case of the three dimensional theory, the dimensionless 't Hooft coupling is $g^2_{S^2}N/\mu$. Using this expression, we see that the dispersion relation agrees with the $k \ll 1$ limit of the one derived in [7] (and reviewed in section 7) from the world-sheet point of view.

The function $h(\lambda)$ cannot be fixed be from the constraints of supersymmetry alone. In the following chapters we determine it for the weakly coupled three dimensional theory at two loop order (up to a single undetermined constant) and at strong coupling from the dual string picture. In the gauge theoretic analysis, we use the known results about the spectrum of the dilatation operator of the four dimensional theory as a benchmark for calibrating our methods and results.

# 4 Weak coupling spectrum and integrable spin chains

The form of the dispersion relation, discussed in the previous section could be determined from an understanding of the realization of the $SU(2|2)$ algebra alone. It has the same "universal" form for all the dimensional reductions of the four dimensional theory whose Hamiltonians are embedded in the $SU(2|2)$ structure as in (20). We shall now focus on the determination of the specific effective Hamiltonians for the gauge theories in question, and show how their spectra are related to those of quantum spin chains. The new results in this section include the determination of $h(\lambda)$ to two loop order in the three dimensional gauge theory, which also appears to be integrable (at least in the $SU(2)$ sector) at this order. We also determine the full $SU(2|3)$ symmetric effective Hamiltonian at one-loop and find its leading correction[6].

To compute the effective Hamiltonians for both the $\mathbb{R}\times S^3$ and $\mathbb{R}\times S^2$ within the scheme of radial quantization we first recall that the states (for both theories) under consideration are generically of the form

$$|i_1 i_2 \cdots i_n\rangle = \frac{1}{\sqrt{N^n}}\,\mathrm{Tr}(a^\dagger_{i_1} a^\dagger_{i_2}\cdots a^\dagger_{i_n})|0\rangle, \qquad a^\dagger_i = (\alpha^\dagger_{4i})_0. \tag{47}$$

$(\alpha^\dagger_{4i})_0$ corresponds to the lowest spherical harmonic mode in the momentum space expansion of the oscillators $(\alpha^\dagger_{4i})$, $i = 1,2,3$ on $S^3$ or $S^2$. Though not displayed above, it is implied that we also include the fermionic states $\psi_\alpha$ so that the Hilbert space transforms under $SU(2|3)$.

As it stands, although these states have a global $SU(N)$ invariance, they do not seem to be invariant under local gauge transformations, which will typically mix the different momentum modes. However, we need to keep in mind that we shall work with a gauge fixed Hamiltonian, for which the states can only be classified by their quantum numbers. In such a gauge fixed $J^{PC}$-like scheme these states are physical and normalizable. In the conformal field theory, these states are mapped to local composite operators built out of scalar fields alone once the theory is mapped to $\mathbb{R}^4$. Local operators with covariant derivatives inserted on $\mathbb{R}^4$ would, in turn, correspond to operators with higher spherical harmonics on $\mathbb{R}\times S^3$. The classification scheme for operators based on $R$ charge and $J^{PC}$ assignments is valid for the three dimensional $\mathcal{N}=8$ theory as well, as is the physicality of the states mentioned above.

At tree level, the Hamiltonians for $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$ and $\mathbb{R}\times S^2$ reduce to harmonic oscillator Hamiltonians, with a single oscillator assigned to each angular momentum mode. The spectrum of the $SU(2|3)$ states is simply given by their engineering dimensions at that level. The one-loop correction to the energies is given by

$$\Delta E^1 = \langle I| : \left(H^4 + H^3\frac{\Pi}{E_0 - H_0}H^3\right) : |I\rangle = \langle I|\Delta H^1|I\rangle, \tag{48}$$

where $H^4$ and $H^3$ are the quartic and cubic parts of the Hamiltonian. $\Pi$ is the projector on to the subspace[7] orthogonal to the states of energy $E_0$. The expressions

[6] For the full $SU(2|3)$ sector, there is an intermediate 'dynamical' contribution between the one and two loop contributions.

[7] Note that this subspace includes states built from non-zero-mode excitations.

for $H^4, H^3$ are also taken to be normal ordered. The normal ordered expressions can be mapped to Hamiltonians of quantum spin chains [14, 15]. The general connection between matrix models and their generalization to field theories and quantum spin chains due to Lee and Rajeev has been reviewed in [16]. For previous use of this identification in context of both the four dimensional gauge theory and the plane wave matrix model we shall refer to [8, 17–19]. Here we recollect some of the relevant facts about these matrix valued operators for the sake of completeness.

The typical term at a given order in perturbation theory takes on the form

$$\Theta^I_J = \frac{1}{N^{(i+j-2)/2}} \operatorname{Tr}\left(W^{\dagger I_1} W^{\dagger I_2} \cdots W^{\dagger I_i} W_{J_j} W_{J_{j-1}} \cdots W_{J_1}\right). \tag{49}$$

The strings $I = I_1 I_2 \cdots I_i$ and $J = J_1 J_2 \cdots J_j$ denote fixed orderings of the bits $I_i, J_j$, etc., which are shorthand for all $SU(2|3)$ and angular momentum indices. The oscillators $W_I$ collectively denote the three bosonic and two fermionic matrix valued oscillator variables. These $SU(N)$ invariant operators form a closed lie super-algebra, whose basic anti-commutation relations are given as

$$\begin{aligned}
[\Theta^I_J, \Theta^K_L]_\pm \;=\;& \delta^K_J \Theta^I_L + \sum_{J=J_1J_2} (-1)^{\epsilon(J_1)[\epsilon(K)+\epsilon(L)]} \delta^K_{J_2} \Theta^I_{J_1 L} \\
&+ \sum_{K=K_1K_2} \delta^{K_1}_J \Theta^{IK_2}_L + \sum_{\substack{J=J_1J_2 \\ K=K_1K_2}} (-1)^{\epsilon(J_1)[\epsilon(K)+\epsilon(L)]} \delta^{K_1}_{J_2} \Theta^{IK_2}_{J_1 L} \\
&+ \sum_{J=J_1J_2} \delta^K_{J_1} \Theta^I_{LJ_2} + \sum_{J=K_1K_2} (-1)^{\epsilon(K_1)[\epsilon(I)+\epsilon(J)]} \delta^{K_2}_J \Theta^{K_1 I}_L \\
&+ \sum_{\substack{J=J_1J_2 \\ K=K_1K_2}} (-1)^{\epsilon(K_1)[\epsilon(I)+\epsilon(J)]} \delta^{K_2}_{J_1} \Theta^{K_1 I}_{LJ_2} \\
&+ \sum_{J=J_1J_2J_3} (-1)^{\epsilon(J_1)[\epsilon(K)+\epsilon(L)]} \delta^K_{J_2} \Theta^I_{J_1 L J_3} \\
&+ \sum_{K=K_1K_2K_3} (-1)^{\epsilon(K_1)[\epsilon(I)+\epsilon(J)]} \delta^{K_2}_J \Theta^{K_1 I K_3}_L \\
&-(-1)^{[\epsilon(I)+\epsilon(J)][\epsilon(K)+\epsilon(L)]} \left(I, J \leftrightarrow K, L\right).
\end{aligned} \tag{50}$$

In the above formula, expressions such as $\sum_{I=I_1I_2}$ imply summing over all ways of writing the string $I$ as the concatenation of two strings $I_1$ and $I_2$. $\epsilon(I)$ denotes the grade of the string $I$, which is zero if it is bosonic and 1 if it is fermionic. The full Lie algebra also includes an ideal which includes elements that encode finite size effects. However, since we shall be working on states of infinite size we shall ignore the contribution of the ideal, which is irrelevant for our present concerns. A more complete discussion of this algebra can be found in [16].

When the operators are bosonic, their action on the single-trace states can be expressed as

$$\Theta^I_J |K\rangle = \delta^K_J |I\rangle + \sum_{K=K_1K_2} \delta^{K_1}_J |IK_2\rangle. \tag{51}$$

Identifying the states with those of a quantum spin chain, we see that $\Theta^{ij}_{ji} = \sum_l P_{l,l+1}$, which is to be replaced by the graded permutation operator $\Pi$ when fermionic creation and annihilation operators are included.

On general grounds of $SU(2|3)$ invariance in the sector of the gauge theories under consideration, the one loop effective Hamiltonian $\Delta H^1 = \alpha\Theta^{ij}_{ji} + \beta\Theta^{ij}_{ij} = \sum_l(\alpha\Pi_{l,l+1} + \beta I_{l,l+1})$, for some constants $\alpha$ and $\beta$. Requiring $\Delta H^1$ to annihilate the chiral primary operators $\mathrm{Tr}(a_3^\dagger)^n|0\rangle$ yields $\alpha = -\beta$. To determine the coefficient of $\Pi$, we see that in the bosonic $SU(2)$ sector the permutation operator arises entirely from the quartic interaction vertex in $H^4$, whose contribution is

$$-\frac{1}{4g^2}\int_\Omega \mathrm{Tr}\left([X_{AB}, X_{CD}][X^{AB}, X^{CD}]\right) \to -\frac{g^2}{|\Omega|\mu^2}\mathrm{Tr}(a_a^\dagger a_b^\dagger a^a a^b) = -\frac{Ng^2}{|\Omega|\mu^2}\sum_l P_{l,l+1}. \tag{52}$$

The formula is true for both $S^2$ and $S^3$, where $|\Omega|$ denotes the associated volume. Substituting the explicit formulae for the volumes, we have

$$\Delta H^1 = \frac{g^2 N\mu}{16\pi^2}\sum_l (I - \Pi_{l,l+1}), \tag{53}$$

for both the theories in the closed $SU(2|3)$ sector. The coupling constant, for the three dimensional theory expressed in terms of $g^2_{S^2}$ is $g^2_{S^2}N/(4\pi)$. It is gratifying to note that for the four dimensional theory, the above formula agrees with the known result for the dilatation operator on $\mathbb{R}^4$ [20], after one sets the radius of $S^3$ to unity, i.e. $\mu = 2$. It also agrees with the one loop result obtained for the three dimensional theory in [21]. In that paper, $\Delta H^1$ was computed for the full $SO(6)$ sector, which is closed, as it is in the four dimensional theory, (only) at one loop. Restriction of that result to the $SU(2)$ sector agrees with above Hamiltonian. Moreover, our understanding of how the $SU(2|3)$ symmetry is realized in the radial Hamiltonian formalism allows us to generalize the one-loop result to the full $SU(2|3)$ sector and, as we shall see below, go beyond the one-loop level.

## 4.1 Higher loops

In [1], it was shown how the $SU(2|3)$ symmetry alone can be used to constrain the form of the higher loop corrections to the dilatation operator of $\mathcal{N} = 4$ SYM. Although the four dimensional superconformal theory was the focus of the analysis in that paper, the results in [1] can be readily adapted to determine the leading corrections to the one-loop radial Hamiltonians for the gauge theories we study as well. Requiring that the generators of the $SU(2|3)$ algebra close order by order in perturbation theory [1] it is possible to write down the complete leading correction to (53) as

$$\Delta H_{SU(2|3)} = \frac{\mu}{2}\left(\lambda(\Theta^{AB}_{AB} - \Theta^{AB}_{BA}) - \sqrt{\frac{(\lambda)^3}{2}}(\epsilon^{\alpha\beta}\epsilon_{abc}\Theta^{abc}_{\alpha\beta} + \epsilon_{\alpha\beta}\epsilon^{abc}\Theta^{\alpha\beta}_{abc})\cdots\right), \tag{54}$$

where the capital indices in the first term on the *r.h.s.* are meant to stand for both the $SU(3)$ (i.e. $a$, $b$, $c$, $\ldots$) and $SU(2)$ ($\alpha$, $\beta$, $\ldots$) indices. The second term breaks

the $SU(2|3)$ invariance to $SU(2) \times SU(3)$ and it encodes the "dynamical" effect of altering the length of the spin chain. It is a non-trivial fact that the effective Hamiltonian is integrable at this order [22]. In the context of the four dimensional gauge theory, we have reproduced the known result for the dilatation generator explicitly within the scheme of radial quantization. However since both the form of $\Delta H_{SU(2|3)}$ given above and its integrability follows directly from the symmetry algebra, we also claim the above formula to be the complete first non-trivial correction to the effective Hamiltonian for the $\mathcal{N} = 8$ model on $\mathbb{R} \times S^2$. Furthermore, it also appears to be integrable at this order.

The coupling constant $\lambda$ is to be identified with $g^2N/(8\pi^2)$ for the four dimensional theory and $g^2_{S^2}N/(2\mu\pi)$ for the $\mathbb{R} \times S^2$ model.

To analyze the question of integrability at two loops it is instructive to restrict ourselves to the bosonic $SU(2)$ sector. For the four dimensional $\mathcal{N} = 4$ theory, the explicit forms of the dilatation operator, are known up to five loop order [1, 23–25]. These results can be readily mapped to the five loop effective radial Hamiltonian using the maps between the 't Hooft couplings of the theories on $\mathbb{R}^4$ and $\mathbb{R} \times S^3$ given before. In the absence of alternate explicit computations of spectra, one can continue to use the symmetry to constrain the radial $SU(2)$ Hamiltonian for the $\mathbb{R} \times S^2$ model at the two loop order up to a single undetermined constant. Stated explicitly

$$\Delta H_{SU(2)} = \frac{\mu}{2}\left(\lambda(\Theta^{ab}_{ab} - \Theta^{ab}_{ba}) + \lambda^2[(2\Theta^{ab}_{ba} - \frac{1}{2}\Theta^{abc}_{cba} - \frac{3}{2}\Theta^{abc}_{abc}) + \alpha_1(\Theta^{ab}_{ab} - \Theta^{ab}_{ba})] + \cdots\right), \tag{55}$$

where $\alpha_1$ is the new undetermined constant for the three dimensional theory which is equal to zero in the four dimensional case, by the requirement of BMN scaling, which is present in the $\mathcal{N} = 4$ theory at this loop order. However, it is not known if there is any reason to expect such a scaling in the three dimensional theory as well. It is known that for the PWMM (as for $\mathcal{N} = 4$ SYM), BMN scaling is violated only at the four-loop order [19, 26]. Nevertheless, the perturbative integrability of the effective Hamiltonian is ensured for arbitrary values of $\alpha_1$. A higher charge $\mathcal{Q} = \lambda\mathcal{Q}_0 + \lambda^2\mathcal{Q}_1$ can be constructed such that $[\Delta H, \mathcal{Q}] = \mathcal{O}(\lambda^4)$. The explicit form of the higher charge is

$$\begin{aligned} \mathcal{Q}_0 &= \Theta^{cab}_{abc} - \Theta^{bca}_{abc}, \\ \mathcal{Q}_1 &= \left(-6\mathcal{Q}_0 + \Theta^{dacb}_{abcd} - \Theta^{bdca}_{abcd} + \Theta^{dbac}_{abcd} - \Theta^{cbda}_{abcd}\right), \end{aligned} \tag{56}$$

and it establishes the two-loop integrability of the $SU(2)$ sector of the three dimensional theory. As we discuss below, the scattering matrix of the spin chain describing the $SU(2|3)$ sector of the gauge theory is factorized, which allows us to interpret the the two loop integrability in the $SU(2)$ sector as an important piece of evidence in favor of integrability of the full $SU(2|3)$ sector at this perturbative order[8]. Finally, we note that at this order, the scaling function is determined to be

$$h^2(\lambda) = 2\lambda + 2\alpha_1\lambda^2 + \cdots . \tag{57}$$

[8]The two-loop Hamiltonian for the full $SU(2|3)$ sector, determined up to a few constants, by requiring the perturbative closure of the algebra is available in [1]. Those results are obviously valid for the three dimensional theory as well.

# 5 Integrability and scattering matrices

In this section we comment on carrying over the insights gained from the studies of the multi-particle $S$-matrix for the planar dilatation operator of $\mathcal{N}=4$ SYM on $\mathbb{R}^4$ to the radial Hamiltonians described in the previous sections. Having interpreted the effective planar Hamiltonians in the $SU(2|3)$ sector of the three and four dimensional gauge theories as spin chains, we can proceed to constrain the $S$-matrix of the spin chain using the symmetry algebra by adapting Beisert's techniques in [2, 3]. We bear in mind that the ferromagnetic vacuum of the spin chains involves states made out of $Z$'s alone, while the excitations/magnons transform under the residual $SU(2|2)$ symmetry, which is also the symmetry of the $S$-matrix. Since the details of the determination of the $S$-matrix by the use of the $SU(2|2)$ algebra have been expanded on at length in [2, 3], we shall refer to Beisert's original papers for the technical details.

The generalization of the single particle (fundamental) representation of the $SU(2|2)$ algebra (38) to multiple particles involves the introduction of the $\mathcal{Z}^{\pm}$ (length changing) markers as follows

$$\begin{aligned}
q^a_\alpha|\phi_b\rangle &= a\,\delta^a_b|\psi_\alpha\rangle,\\
q^{*\alpha}_a|\phi_b\rangle &= c\,\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\beta\,\mathcal{Z}^-\rangle,\\
q^a_\alpha|\psi_\beta\rangle &= b\,\epsilon_{\alpha\beta}\epsilon^{ab}|\phi_b\,\mathcal{Z}^+\rangle,\\
q^{*\alpha}_a|\psi_\beta\rangle &= d\,\delta^\alpha_\beta|\phi_a\rangle.
\end{aligned}\tag{58}$$

Comparison with (20) immediately clarifies the rôle of the markers as essentially bookkeeping devices for gauge transformations. The generators $P$ and $K$ act as

$$P|W\rangle = ab|W\mathcal{Z}^+\rangle,\quad K|W\rangle = cd|W\mathcal{Z}^-\rangle.\tag{59}$$

In the case of the scattering of two magnons, the two particle $S$-matrix can be constrained up to ten undetermined functions of the magnon momenta. The action of the $S$-matrix on two particle states can be expressed in all generality as

$$\begin{aligned}
S_{12}|\phi^1_a\phi^2_b\rangle &= A_{12}|\phi^2_{\{a}\phi^1_{b\}}\rangle + B_{12}|\phi^2_{[a}\phi^1_{b]}\rangle + \frac{1}{2}C_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\psi^2_\alpha\psi^1_\beta\mathcal{Z}^-\rangle\\
S_{12}|\psi_\alpha\psi_\beta\rangle &= D_{12}|\psi^2_{\{\alpha}\psi^1_{\beta\}}\rangle + E_{12}|\psi^2_{[\alpha}\psi^1_{\beta]}\rangle + \frac{1}{2}F_{12}\epsilon_{\alpha\beta}\epsilon^{ab}|\phi^2_a\phi^1_b\mathcal{Z}^+\rangle\\
S_{12}|\phi^1_a\psi^2_\beta\rangle &= G_{12}|\psi^2_\beta\phi^1_a\rangle + H_{12}|\phi^2_a\phi^1_\alpha\rangle\\
S_{12}|\psi^a_\alpha\phi^2_b\rangle &= K_{12}|\psi^2_\alpha\phi^1_b\rangle + L_{12}|\phi^2_b\psi^1_\alpha\rangle
\end{aligned}$$

Requiring the two body scattering matrix to commute with the supersymmetry generators uniquely determines the ten undetermined functions in terms of a single function $S^0_{12}$. For example

$$A_{12} = S^0_{12}\frac{x^+_2 - x^-_1}{x^-_2 - x^+_1}.\tag{60}$$

The expressions for all the other functions $B\cdots L$ in terms of $S^0_{12}$ can be found in table-1 of [2]. Furthermore, the scattering matrix satisfies the Yang-Baxter algebra

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12},\tag{61}$$

fulfilling the necessary condition for the integrability of the $SU(2|3)$ symmetric spin chain to all orders in perturbation theory[9]. The magnon momenta are to be determined by the Bethe ansatz equations obeyed by the scattering matrix [2, 3]. For an $m$ magnon state, the total energy is given by the additive relation

$$H = \sum_{i=1}^{m} H_i = \sum_{i=1}^{m} \frac{\mu}{2}\sqrt{1 + 4h^2(\lambda)\sin^2(k_i/2)}. \qquad (62)$$

The factorizability of the $S$-matrix and its determination up to a single function are both consequences of the fact that the underlying symmetry is $SU(2|2)$ and that the fundamental excitations fall on atypical representations. The tensor product of this representation uniquely gives a *single* new irreducible representation, allowing us to constrain the scattering matrix up to a single function of the magnon momenta. The formal (matrix) structure of the two particle scattering matrix, is hence a direct consequence of the symmetry and the details of the models that realize these symmetries lie hidden in the parameter $\mu$, the scaling function $h(\lambda)$ and, relatedly, $S^0_{12}$.

While the Yang-Baxter condition on the $S$-matrix is a necessary condition for integrability, it is certainly not sufficient. One typically needs to augment this with the existence of additional conserved charges to gain surer evidence of integrability. In the case of the three dimensional gauge theory we have presented this evidence in the form of higher conserved charges up to the two loop order. At least in the $SU(2)$ sector, one can hope to reliably use the asymptotic Bethe ansatz techniques to compute the spectrum of the three dimensional gauge theory at this loop order. The Yang-Baxter relation satisfied by the full $SU(2|2)$ S-matrix suggests that Bethe ansatz techniques may be applicable to a larger sector of the gauge theory at and beyond two loops. Clearly, probing the structure of the spin chain describing its spectrum at higher loop orders (even in the $SU(2)$ sector) and understanding its integrability properties remains an exciting open problem.

# 6 Comments on PWMM and $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$

The principles used in constraining the spectrum of the three dimensional gauge theory used above can also be employed in the study of the PWMM and $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$. These theories have a rich moduli space of vacua. However, as long as a well defined large-$N$ expansion can be implemented, the spectra around each of those vacua can, in principle, be constrained by exactly the same use of the $SU(2|2)$ algebra as explained above. The different vacua would simply correspond to different $h(\lambda)$. For example, the $h$ function for the PWMM around its trivial vacuum has been computed at the leading order on the string theory side in [7] and up to the four loop level in perturbation theory in [19][10]. A different large-$N$

[9]The Yang-Baxter algebra is also a consequence of the Yangian symmetry exhibited by the $S$-matrix [27].

[10]For an exposition of the perturbative realization of $SU(2|2)$ in weaky coupled PWMM see [28].

limit for the matrix model can also be taken if it is expanded around its so-called fuzzy-sphere vacuum [29]. This expansion simply maps the matrix model to the three dimensional gauge theory studied above, and the corresponding $h$ would be the one computed up to two loops in the preceding section. Thus, while the different $h$ functions are determined dynamically in these different theories, the rôle of the underlying $SU(2|2)$ and the consequent dispersion relation (62) appears to be generic to the dimensional reductions of $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$ that preserve the $SU(2|3)$ symmetry. Furthermore, the "matrix" structure of the $SU(2|2)$ S-matrix would also be the same for this class of theories, with the specifics of the models and choices of vacua being encoded in the dynamically determined observable $A_{12}$. As mentioned before, these universal properties are however crucially contingent on there being a systematic large-$N$ limit for the study of the spectrum of any of these given models around a particular vacuum, which is assumed to be "well separated" in the sense described below[11].

# 7 $SU(2|2)$ from string theory

In this section we will discuss the emergence of the $SU(2|2)$ algebra in the string theories dual to the family of gauge theories with 16 supercharges discussed previously. We will succeed in deriving the explicit algebra in the plane-wave (BMN-like) limit, i.e. in the limit that the magnon momenta $k \ll 1$. This amounts to analyzing type IIA strings on the plane-wave, for which exact quantization has been performed. The main point of the analysis is to show the emergence of the central charges through a relaxation of the level-matching condition. This was originally understood for the full $AdS_5\times S^5$ superstring in [11]; we will have to content ourselves with the plane-wave limit, as the full geometries relevant to our cases are complicated and have so far not admitted a solvable sigma model. We find it useful to first review the bubbling geometries method which was used to construct the full dual supergravity solutions – the so-called Lin-Maldacena solutions – in which the problem is reduced to a classical axisymmetric electrostatics problem [7]. The results presented here and in sections 7.1 and 7.2 are known from the literature. Our contribution – deriving the $SU(2|2)$ algebra – is presented in section 7.3.

The geometries have the bosonic symmetry $\mathbb{R}\times SO(3)\times SO(6)$; they contain (in addition to the temporal direction) an $S^2$ and an $S^5$ whose radii vary with the remaining two coordinates $\rho$ and $\eta$. The geometries may be thought of as arising from M2 and M5 branes wrapping these contractile spheres. These geometries also contain a $(\rho,\eta)$-dependent dilaton and B-field, of which the latter has its legs in the $S^2$. There are also one-form and three-form Ramond-Ramond potentials $C_1$ and $C_3$, similarly dependent only on $(\rho,\eta)$, for which $C_1\propto dt$ and $C_3\propto dt\wedge d\Omega_2$, where $d\Omega_2$ denotes again the $S^2$. The $(\rho,\eta)$ plane is the scene of the axisymmetric electrostatics problem ($\rho$ is the radial, and $\eta$ the axial coordinate), whereby the electric potential $V(\rho,\eta)$ comes to inform the specific dependence of the geometry on those coordinates.

[11] The well-separatedness of the vacua and formulations of large-$N$ limits are not independent issues [7, 29].

More specifically the electric potential is that found in the presence of a configuration of "critically" charged conducting disks centered on the $\eta$-axis, subject to a certain asymptotically defined external electric field. The "critical" charge corresponds to the condition that the charge density exactly vanishes at the edge of the disks, which in turn implies that the electric field is non-infinite there. This condition on the charge, and the asymptotic form of the external potential, are determined by requiring that the corresponding supergravity solutions are well-behaved and non-singular. The total charge per disk and the distance between disks are proportional to the units of $*dC_3$ and $dB_2$ flux on six- and three-cycles constructed from the $S^5$ and $S^2$ and a $\rho$ or, respectively, $\eta$ fiber. The emerging picture is very attractive, with the solutions in one-to-one correspondence with disk configurations.

The disk configurations corresponding to the dual of SYM on $\mathbb{R} \times S^2$ are the simplest: finite radii disks, a single disk corresponding to the trivial vacuum where all scalar fields have zero VEV. Adding more disks corresponds to the other vacua of the theory, see [7] for a discussion. SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$ consists of a periodic extension of this configuration, extending to $\pm\infty$ in $\eta$. Finally, the dual of the PWMM is obtained from an infinite disk at $\eta = 0$ giving the trivial vacuum, other vacua being obtained through the addition of finite-radii disks at $\eta > 0$. The simplest plane-wave limit of these geometries is given by the IIA plane-wave with $SO(3) \times SO(4)$ symmetry

$$\begin{aligned} ds^2 &= -2dx^+dx^- - \left(\left(\frac{\beta}{3}\right)^2 x^i x^i + \left(\frac{\beta}{6}\right)^2 x^{i'} x^{i'}\right)(dx^+)^2 + dx^i dx^i + dx^{i'} dx^{i'}, \\ F_{+123} &= \beta = -3F_{+4}, \end{aligned} \tag{63}$$

where $i$ and $i'$ run from 1 to 4 and 5 to 8, respectively, and where $\beta$ is an arbitrary positive constant. This geometry is obtained by expanding the Lin-Maldacena geometries around the region corresponding to the edge of a single disk, ensuring any other disks are well-separated from this region; this is what is meant by "well-spaced" vacua in footnote 4.

Consistent with the focus of the paper, we will use SYM on $\mathbb{R} \times S^2$ around the trivial vacuum as the prototypical example, but of course the analysis following (in sections 7.1 - 7.4) is valid for the single-disk, plane-wave limit dual of any of the gauge theories. The full supergravity solution dual to the trivial vacuum of SYM on $\mathbb{R} \times S^2$ is [7]

$$\begin{aligned} ds^2_{LM} &= \alpha' L^{1/3}\Big[-8(1+r^2) f dt^2 + 16 f^{-1}\sin^2\theta d\Omega_5^2 \\ &\quad + \frac{8rf}{r+(1+r^2)\arctan r}\left(\frac{dr^2}{1+r^2} + d\theta^2\right) + \frac{2r\left[r+(1+r^2)\arctan r\right]f}{1+r\arctan r} d\Omega_2^2\Big], \\ f &\equiv \sqrt{\frac{2}{r}[r+(\cos^2\theta + r^2)\arctan r]}, \\ B_2 &= -L^{1/3}\frac{2\sqrt{2}\left[r+(-1+r^2)\arctan r\right]\cos\theta}{1+r\arctan r} d^2\Omega, \\ e^\Phi &= K L^{1/2} 8 r^{\frac{1}{2}}(1+r\arctan r)^{-\frac{1}{2}}[r+(1+r^2)\arctan r]^{-\frac{1}{2}} f^{-\frac{1}{2}}, \end{aligned} \tag{64}$$

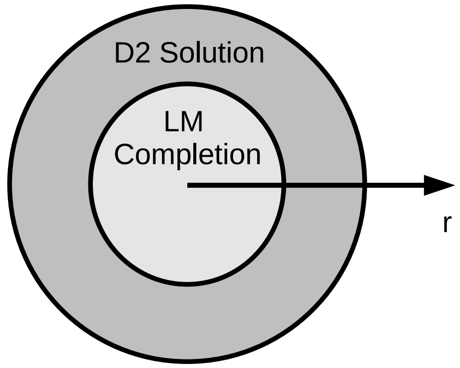


Figure 1: The Lin-Maldacena solution dual to SYM on $\mathbb{R} \times S^2$ is a completion of the D2-brane geometry [30] to the IR region (small $r$).

with

$$C_1 = -K^{-1} L^{-\frac{1}{3}} \frac{[r + (1+r^2)\arctan r]\cos\theta}{2r} dt,$$
$$C_3 = -K^{-1} \frac{r[r + (1+r^2)\arctan r]^2 f^2}{\sqrt{2}(1 + r\arctan r)} dt \wedge d^2\Omega, \tag{65}$$

where $L$ and $K$ are constants which will be related to gauge theory parameters below. This solution may be viewed as an IR completion of the D2-brane solution on $\mathbb{R} \times S^2$ [30], which suffers from a diverging dilaton as the radial coordinate $r$ approaches zero, invalidating the 10-dimensional description, see figure 1. The D2-brane solution is given by

$$ds^2_{D2} = \alpha' C^{2/3} \left( r^{5/2}\left(-dt^2 + d\Omega_2^2\right) + \frac{dr^2}{r^{5/2}} + r^{-1/2} d\Omega_6^2 \right), \quad e^\Phi = \frac{g^2}{\mu C^{1/3}} r^{-5/4}, \tag{66}$$

where $C^2 = 6\pi^2 g^2 N/\mu$, $g$ is the Yang-Mills coupling constant, and $1/\mu$ is the radius of the gauge-theory $S^2$. Here we have used dimensionless coordinates $t$ and $r$ in order to match-up with those of $ds^2_{LM}$. It will be important for us later in matching to the gauge theory results that $t = \mu\, t_{YM}$, where $t_{YM}$ is the dimensionful gauge theory time coordinate[12]. Indeed taking $r \to \infty$ and scaling $r \to (8/\pi)^{1/3} r$, $t \to t/2$ in $ds^2_{LM}$, $ds^2_{D2}$ is recovered with the following identification of parameters

$$L = \frac{3\pi^3\lambda}{2^9\sqrt{2}}, \quad K = \frac{4g^2\sqrt{2}}{(6\lambda)^{2/3}\pi}, \quad t_{LM} = \frac{t_{D2}}{2} = \frac{\mu}{2} t_{YM}, \tag{67}$$

where we have introduced the dimensionless gauge theory 't Hooft coupling $\lambda = g^2 N/\mu$. As was explained in [7], this identification gives $L$ and $K$ in terms of gauge theory quantities. Plugging them into the Lin-Maldacena solution string coupling $\exp(\Phi)$, one finds $\sim \lambda^{5/6}/N \times g(r,\theta)$, where the function $g(r,\theta)$ is always finite (i.e.

[12]The coordinate $r$ is related to the usual coordinate $U$ from [30] by $U = C^{2/3}\mu r$.

$\leq \mathcal{O}(1)$) and goes to zero for large-$r$ as $r^{-5/4}$. This implies that at large-$N$, the string coupling is suppressed everywhere, and so the solution may be trusted for any $r$.

The coordinate $r$ is related to the gauge theory energy scale. The running of the SYM coupling constant is trivial and given simply by dimensional analysis, so that the dimensionless effective coupling is given by $g_{eff} = g^2 N/E$, where $E$ is the relevant energy scale. Since we have SYM on $\mathbb{R} \times S^2$, it is more sensible to express this scale in units of the $S^2$ radius $1/\mu$, i.e. $\mathcal{E} = E/\mu$, and so write $g_{eff} = \lambda/\mathcal{E}$. This running is reflected in the string solution by the coordinate dependence of the string coupling $\exp(\Phi)$. This allows us to identify $\mathcal{E} \sim g^{-6/5}(r,\theta)$ and so at large-$r$, $\mathcal{E} \sim r^{3/2}$. The curvature scale of the geometry, for example the inverse radius of the $S^5$, diverges for large-$r$. Thus the strongly curved region corresponds to weak effective gauge coupling or large gauge theory energies; we will therefore call this the UV region. In this part of the geometry (for large-$N$), strings propagate classically on a string-scale-curved background. The small-$r$ or IR region corresponds to strong effective gauge theory coupling and weak curvature scales in the geometry, where a classical supergravity analysis is appropriate.

If the large-$N$ limit is relaxed, the geometry may still be trusted for large enough $r$, but at small $r$ the string coupling will be large leading to a transition to an 11-dimensional description. For small $N$, one expects then to make contact with the superconformal M2-brane or Bagger-Lambert-Gustavsson [31–34] theory (i.e. ABJM [35] at $k = 1, 2$) and their massive counterparts [36–39]. Of course, we do not expect to find integrability at finite $N$.

We are interested in taking a Penrose limit of the Lin-Maldacena geometry around a stable light-like geodesic on the $S^5$. The geodesic is a line in the $t$-$\phi$ plane, where $\phi$ describes a great circle in the $S^5$. This geodesic is located in the deep IR at $r = 0$, $\theta = \pi/2$, corresponding to the strongly coupled limit of the gauge theory[13]. The radius of the $S^5$ in (64) at $r = 0$ and $\theta = \pi/2$ is equal to

$$\frac{R^2_{S^5}}{\alpha'} = 8\sqrt{2} L^{1/3} = \left(6\pi^3\lambda\right)^{1/3}, \tag{68}$$

where we have used (67) to express it in terms of gauge theory quantities. In order to take the Penrose limit we define the energy $E$ and angular momentum $J$ generators as $i\partial_t \equiv E - J$ and $-i\partial_\phi \equiv J$, and then define ($R^2 = R^2_{S^5}$)

$$\begin{aligned} x^+ &= t, \quad x^- = R^2(t-\phi), \quad p^- = i\partial_{x^+} = i\partial_t = E - J, \\ p^+ &= i\partial_{x^-} = R^{-2}(i\partial_t - i\partial_\phi) = R^{-2}E \simeq R^{-2}J, \end{aligned} \tag{69}$$

so that the light-cone energy is zero for $E = J$. We will take the usual BMN-like limit by taking $R \to \infty$ and concentrating on states with finite $p^+$ and $p^-$, so that

[13]This location in the $r$-$\theta$ plane corresponds to $|g_{tt}| = g_{\phi\phi}$. At this location we have a massless geodesic corresponding to $E = J$. Away from $r = 0$, $\theta = \pi/2$, $E > J$ and the geodesic describes a particle of mass $m^2 = E^2 - J^2$. Thus the chosen geodesic is a stable minimization of the energy [7].

$E \gtrsim J \sim R^2$. Starting from (64) we take a Penrose limit around $r = 0$, $\theta = \pi/2$

$$\begin{aligned} \theta &= \frac{\pi}{2} + \frac{\sqrt{2}}{R}\, x^{i=1}, \quad r\Theta^{i=2,3,4} = \frac{\sqrt{2}}{R}\, x^{i=2,3,4}, \\ d\Omega_5^2 &= \frac{dy^2}{(1+y^2/4)^2} + \frac{(1-y^2/4)^2}{(1+y^2/4)^2} d\phi^2, \quad y^{i'} = \frac{x^{i'}}{R}, \end{aligned} \tag{70}$$

where $\vec{\Theta}$ is the embedding of the unit-$S^2$ which appears as $d\Omega_2^2$ in (64) into $\mathbb{R}^3$. This gives the IIA plane-wave

$$ds^2 = -2dx^+dx^- - \left(4x^ix^i + x^{i'}x^{i'}\right)(dx^+)^2 + dx^idx^i + dx^{i'}dx^{i'} + \mathcal{O}(R^{-2}), \tag{71}$$

where one can match-up with (63) by setting[14] $\beta = 6$. In the following sections we will analyze the $x^{i'}$ excitations of strings on this geometry. It is of course well-known that the energy of such excitations are given by

$$p^- = \sum_i \sqrt{\left(\frac{\beta}{6}\right)^2 + \frac{n_i^2}{(\alpha' p^+)^2}}, \tag{72}$$

where $n_i$ are the worldsheet momenta of the excitations. Using (67) and (69), and setting $\beta = 6$ one obtains [7]

$$H_{YM} = \frac{\mu}{2}\, p^- = \frac{\mu}{2} \sum_i \sqrt{1 + \frac{R^4}{\alpha'^2} \frac{n_i^2}{J^2}}, \tag{73}$$

which matches the gauge theory result (62) in the $k_i = n_i/J \ll 1$ limit, and through (68), gives us the strong coupling limit of the function $h(\lambda)$

$$h(\lambda) \simeq \left(6\pi^3\lambda\right)^{1/3}. \tag{74}$$

It is worth pointing out that while the three dimensional gauge theory only has sixteen supersymmetries the plane wave geometry is invariant under 24 supersymmetries. This is suggestive of an enhancement of supersymmetry in the BMN limit of the strongly coupled gauge theory and a potential connection between the three dimensional strongly coupled SYM theory and $\mathcal{N} = 6$ Chern-Simons models.

We will now continue with an analysis of the supersymmetry algebra of strings on the IIA plane-wave, with the ultimate goal of uncovering the $SU(2|2)$ structure found in the gauge theory. We begin with some general known results.

[14] One may also verify that the Ramond-Ramond field strengths come out correctly. Note that the constant $\beta$ may be absorbed into the coordinates and their relations to gauge theory parameters and has no physical significance in the gauge theory.

### 7.1 Light-cone gauge strings on a plane-wave

The light-cone gauge quantization of strings on the plane-wave geometry (63) corresponding to the Penrose limit described in the previous section was carried out in the series of papers [40–42]. Using this analysis, we will show that the $SU(2|2)$ algebra emerges from the commutation relations of the supercharges. The main issue is to show that the central charges emerge from a relaxation of the level-matching condition, exactly as was shown for strings on $AdS_5 \times S^5$ in [11]. We will find it useful to repeat portions of the analysis in [41], both in the interest of readability and because we will use slightly different conventions.

The string action is given by

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \Big\{ \sqrt{-h}h^{ab}\Big[ &-2\partial_a X^+ \partial_b X^- + \partial_a X^I \partial_b X^I - M(X^I)\partial_a X^+ \partial_b X^+ \\ &- 2\partial_a X^+ \bar{\theta}\Gamma^- \partial_b \theta + \partial_a X^+ \partial_b X^+ \Upsilon(\theta)\Big] + 2\epsilon^{ab}\, \partial_a X^+ \, \bar{\theta}\Gamma^{-9}\partial_b\theta \Big\}, \end{aligned} \tag{75}$$

where the index $I = (i, i')$, where $i = 1, \ldots, 4$ and $i' = 5, \ldots, 8$, and where we have introduced the shorthand

$$M(X^I) = \left(\frac{\beta}{3}\right)^2 X^i X^i + \left(\frac{\beta}{6}\right)^2 X^{i'} X^{i'}, \qquad \Upsilon(\theta) = \frac{\beta}{2}\bar{\theta}\Gamma^- \left(\Gamma^{123} + \frac{1}{3}\Gamma^{49}\right)\theta. \tag{76}$$

We will hold off for the time being on the explicit details of the fermions $\theta$ and the 10-d gamma matrices $\Gamma^A$. We continue by calculating the Virasoro constraints, and then imposing the light-cone gauge

$$h_{ab} = \mathrm{diag}(-1, 1), \quad X^+ = \alpha' p^+ \tau. \tag{77}$$

We find

$$\begin{aligned} X^{-\prime} &= \frac{1}{\alpha' p^+}\left(\dot{X}\cdot X' - \alpha' p^+ \bar{\theta}\Gamma^- \theta'\right), \\ \dot{X}^- &= \frac{1}{2\alpha' p^+}\left(\dot{X}^2 + X'^2 - (\alpha' p^+)^2 M(X^I) - 2\alpha' p^+ \bar{\theta}\Gamma^- \dot{\theta} + (\alpha' p^+)^2 \Upsilon(\theta)\right), \end{aligned} \tag{78}$$

where $\dot{} = \partial_0 = \partial_\tau$ and $' = \partial_1 = \partial_\sigma$, and the inner product is over the composite $I$ index. Using these expressions to eliminate $X^-$ we may calculate the light-cone conjugate momenta. The Lagrangian $L$ and Lagrangian density $\mathcal{L}$ are given as

$$S = \int d\tau\, L = \int d^2\sigma\, \mathcal{L}, \tag{79}$$

from which we calculate

$$\begin{aligned} p^+ &= \int_0^{2\pi} d\sigma\, P^+ = \int_0^{2\pi} d\sigma \left(-\frac{\delta\mathcal{L}}{\delta \dot{X}^-}\right), \\ p^- &= \int_0^{2\pi} d\sigma\, P^- = \int_0^{2\pi} d\sigma \left(-\frac{\delta\mathcal{L}}{\delta \dot{X}^+}\right) \equiv H. \end{aligned} \tag{80}$$

The first of these equalities is a tautology resulting from the consistent definition of $X^+$, while the second yields the light-cone Hamiltonian $H$, given by

$$H = \frac{1}{4\pi\alpha'^2 p^+}\int_0^{2\pi} d\sigma\left(\dot{X}^2 + X'^2 + (\alpha' p^+)^2 M(X^I) + 2\alpha' p^+ \bar\theta\Gamma^{-9}\theta' - (\alpha' p^+)^2 \Upsilon(\theta)\right). \quad (81)$$

The first Virasoro constraint in (78) yields one extra piece of information, the level matching condition. It is this condition which is relaxed in order to reveal the central charge of the $SU(2|2)$ algebra. This issue has been worked out for the $AdS_5 \times S^5$ string in [11], where it has been shown that the central charge, related as it is to changing the length of the gauge theory spin-chain [2], appears in the string treatment by going off-shell through a relaxation of the level matching condition. Specifically, one takes the total worldsheet momentum $p_{ws}$ to be

$$p_{ws} = \frac{1}{\alpha' p^+}\int_0^{2\pi} d\sigma\left(\dot{X}\cdot X' - \alpha' p^+ \bar\theta\Gamma^- \theta'\right) \neq 0. \quad (82)$$

We will see that the supersymmetry algebra will allow us to associate $p_{ws}$ with the gauge theory magnon momentum $k$.

## 7.2 Supercharges and algebra

The fermions $\theta$ are given by [42]

$$\theta = \frac{1}{\sqrt{2\alpha' p^+}}\frac{1}{2^{1/4}}\begin{pmatrix} 0 \\ \psi^A \end{pmatrix}, \quad (83)$$

where the $\psi^A$ are 16-component real and are further decomposed according to their $SO(8)$ and $SO(4)$ chiralities

$$\gamma^9\psi^1_\pm = +\psi^1_\pm, \quad \gamma^9\psi^2_\pm = -\psi^2_\pm, \quad \gamma^{1234}\psi^A_\pm = \pm\psi^A_\pm. \quad (84)$$

The gamma matrices and related conventions are collected in appendix A. There are 2 dynamical supercharges $Q^1_+$ and $Q^2_-$ which have been constructed in [41], they are

$$\begin{aligned} Q^1_+ &= \frac{1}{4\pi\alpha'}\frac{1}{\sqrt{\alpha' p^+}}\int_0^{2\pi} d\sigma\Big(\partial_- X^i\gamma^i\psi^1_- + \frac{m}{3}X^i\gamma^i\gamma^4\psi^2_+ \\ &\qquad + (i\to i',\ \psi_+ \leftrightarrow \psi_-,\ m\to -m/2)\Big), \\ Q^2_- &= \frac{1}{4\pi\alpha'}\frac{1}{\sqrt{\alpha' p^+}}\int_0^{2\pi} d\sigma\Big(\partial_+ X^i\gamma^i\psi^2_+ - \frac{m}{3}X^i\gamma^i\gamma^4\psi^1_- \\ &\qquad + (i\to i',\ \psi_+ \leftrightarrow \psi_-,\ m\to -m/2)\Big), \end{aligned} \quad (85)$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$, and $m = \beta\alpha' p^+$. The canonical commutation relations for the fields at equal times $\tau$ are given by

$$[X^I(\sigma), \dot{X}^J(\sigma')] = i2\pi\alpha'\delta^{IJ}\delta(\sigma-\sigma'), \quad \{\psi^A_\pm(\sigma), \psi^B_\pm(\sigma')\} = 2\pi\alpha'\delta^{AB}\delta(\sigma-\sigma'). \quad (86)$$

In order to reveal the $SU(2|2)$ algebra we are interested in, we find it necessary to define the following projected supercharges

$$Q=\frac{e^{i\pi/4}}{\sqrt{2}}(1+\gamma^4)\left(Q^1_+ + iQ^2_-\right), \qquad \bar{Q}\equiv Q^*=\frac{e^{-i\pi/4}}{\sqrt{2}}(1+\gamma^4)\left(Q^1_+ - iQ^2_-\right), \quad (87)$$

and then to restrict $Q$ and $\bar{Q}$ to the appropriate subalgebra. First let us quote the result before the restriction. We find[15]

$$\begin{aligned}
\{Q_\alpha,\bar{Q}_\beta\} &= (1+\gamma^4)_{\alpha\beta}\,H+\frac{\beta}{3}\,\mathcal{J}_{\alpha\gamma}(1+\gamma^4)_{\gamma\beta}-\frac{\beta}{6}\,\mathcal{J}'{}_{\alpha\gamma}(1+\gamma^4)_{\gamma\beta},\\
\{Q_\alpha,Q_\beta\} &= -i\frac{p_{ws}}{2\pi\alpha'}(1+\gamma^4)_{\alpha\beta},\\
\{\bar{Q}_\alpha,\bar{Q}_\beta\} &= i\frac{p_{ws}}{2\pi\alpha'}(1+\gamma^4)_{\alpha\beta},
\end{aligned} \qquad (88)$$

where $\alpha,\beta,\gamma$ are $SO(8)$ spinor indices ranging from 1 to 16, and where

$$\begin{aligned}
\mathcal{J}_{\alpha\gamma} &= \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\Big(\dot{X}^{\hat{i}}X^{\hat{j}}-\dot{X}^{\hat{j}}X^{\hat{i}}\\
&\qquad -\frac{i}{4}\left(\psi^1_-\gamma^{\hat{i}\hat{j}}\psi^1_- +\psi^1_+\gamma^{\hat{i}\hat{j}}\psi^1_+ +\psi^2_+\gamma^{\hat{i}\hat{j}}\psi^2_+ +\psi^2_-\gamma^{\hat{i}\hat{j}}\psi^2_-\right)\Big)\,\gamma^{\hat{i}\hat{j}}_{\alpha\gamma},\\
\mathcal{J}'{}_{\alpha\gamma} &= \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\Big(\dot{X}^{i'}X^{j'}-\dot{X}^{j'}X^{i'}-\frac{i}{4}\left(\psi^1_-\gamma^{i'j'}\psi^1_- +\psi^2_+\gamma^{i'j'}\psi^2_+ +\right)\Big)\,\gamma^{i'j'}_{\alpha\gamma},
\end{aligned} \qquad (89)$$

where we have introduced the $SO(3)$ index $\hat{i}=1,2,3$. Notice the crucial observation that the centrally extended algebra (i.e. non-zero values for the $\{Q,Q\}$ and $\{\bar{Q},\bar{Q}\}$ commutators) comes from a relaxation of the level-matching condition (82).

## 7.3 Restriction to $SU(2|2)$

In order to uncover the $SU(2|2)$ structure, we need to decompose the $SO(8)$ fermions into $SU(2)^4$. This decomposition is discussed in detail in [43]. We note that $\psi^2_\pm$ is in the $\mathbf{8_c}$ of $SO(8)$ while $\psi^1_\pm$ is in the $\mathbf{8_s}$. The decomposition into $SU(2)^4$ is different for the different $SO(8)$ chiralities

$$\begin{aligned}
&\mathbf{8_s}\to(\mathbf{2},\mathbf{2})\oplus(\mathbf{2'},\mathbf{2'}), \quad \text{i.e. } \psi^1_a\to\psi^1_{+\alpha_1\alpha_2}\oplus\psi^{1\,\dot{\alpha}_1\dot{\alpha}_2}_-,\\
&\mathbf{8_c}\to(\mathbf{2},\mathbf{2'})\oplus(\mathbf{2'},\mathbf{2}), \quad \text{i.e. } \psi^2_{\dot{a}}\to\psi^{2\,\dot{\alpha}_2}_{+\alpha_1}\oplus\psi^{2\,\dot{\alpha}_1}_{-\alpha_2},
\end{aligned} \qquad (90)$$

where $a$ and $\dot{a}$ run from 1 to 8 and $\alpha_1,\alpha_2,\dot{\alpha}_1,\dot{\alpha}_2$ are the indices of the four $SU(2)$'s. The indices and gamma matrices are expounded in appendix A. We are interested in excitations lying in the $SO(4)$ piece of the geometry (i.e. labelled by indices $i',j'$). Therefore we restrict our attention to the $X^{i'}$ fields and their superpartners $\psi^2_-$ and

[15] We also find a contribution to $\{Q,Q\}$ and $\{\bar{Q},\bar{Q}\}$ given by $-\frac{i}{8\pi\alpha'}\frac{\beta}{3}\int_0^{2\pi}d\sigma\left(\psi^1_-\psi^1_- -\psi^1_+\psi^1_+ +\psi^2_+\psi^2_+ -\psi^2_-\psi^2_-\right)(1+\gamma^4)_{\alpha\beta}$, which is a vanishing sum of $\delta(0)$ infinities.

$\psi^1_+$. There is a freedom in choosing either $Q \sim Q^1_+ + i\gamma^4 Q^2_-$ or $Q \sim \gamma^4 Q^1_+ + iQ^2_-$ for our $SU(2|2)$ supercharge[16], without loss of generality, we choose the latter option. Specifically, we define

$$
\begin{aligned}
Q^{\dot\alpha_2}_{\dot\alpha_1} &= \frac{e^{i\pi/4}}{4\pi\alpha'}\frac{1}{\sqrt{\alpha' p^+}}\int_0^{2\pi} d\sigma \Big\{ \Big(i\partial_+ + \frac{m}{6}\Big) X^{\dot\alpha_2\gamma_2}\psi^2_{-\dot\alpha_1\gamma_2} \\
&\qquad\qquad + \Big(i\partial_- + \frac{m}{6}\Big) X^{\dot\alpha_2\gamma_2}\, i\sigma^{4\sigma_1}_{\dot\alpha_1}\,\psi^1_{+\sigma_1\gamma_2}\Big\}, \\
\bar Q^{\dot\alpha_1}_{\dot\alpha_2} &= \frac{e^{-i\pi/4}}{4\pi\alpha'}\frac{1}{\sqrt{\alpha' p^+}}\int_0^{2\pi} d\sigma \Big\{ \Big(-i\partial_+ + \frac{m}{6}\Big) X^{\gamma_2}_{\dot\alpha_2}\psi^{2\,\dot\alpha_1}_{-\gamma_2} \\
&\qquad\qquad - \Big(-i\partial_- + \frac{m}{6}\Big) X^{\gamma_2}_{\dot\alpha_2}\, i\sigma^{4\sigma_1\dot\alpha_1}\,\psi^1_{+\sigma_1\gamma_2}\Big\},
\end{aligned}
\tag{91}
$$

where we have defined $X^{\dot\alpha_2\gamma_2} = X^{i'}\sigma^{i'\,\dot\alpha_2\gamma_2}$. We will now express these supercharges in terms of string oscillators. We will be interested in the action of the algebra on excited states, and so we leave out the zero-mode part of the following expressions. The mode expansions for the fields were worked out in detail in [41] and are collected in appendix A. With these expansions we find

$$
\begin{aligned}
Q^{\dot\alpha_2}_{\dot\alpha_1} &= i\frac{e^{i\pi/4}}{\sqrt{2\alpha' p^+}}\sum_{n\neq 0}\Omega_n\left(\alpha_n^{\dot\alpha_2\gamma_2}\psi_{-n\,\dot\alpha_1\gamma_2} + \tilde\alpha_n^{\dot\alpha_2\gamma_2}\tilde\psi^4_{-n\,\dot\alpha_1\gamma_2}\right), \\
\bar Q^{\dot\alpha_1}_{\dot\alpha_2} &= -i\frac{e^{-i\pi/4}}{\sqrt{2\alpha' p^+}}\sum_{n\neq 0}\Omega_{-n}\left(\alpha^{\gamma_2}_{n\,\dot\alpha_2}\psi_{-n\,\gamma_2}^{\;\;\dot\alpha_1} - \tilde\alpha^{\gamma_2}_{n\,\dot\alpha_2}\tilde\psi^{4\,\dot\alpha_1}_{-n\,\gamma_2}\right),
\end{aligned}
\tag{92}
$$

where $\tilde\psi^4_{-n\,\dot\alpha_1\gamma_2} = i\,\sigma^{4\,\alpha_1}_{\dot\alpha_1}\tilde\psi_{-n\,\alpha_1\gamma_2}$, and where

$$
\Omega_n = \frac{1 + \frac{6}{m}(\omega_n - n)}{\sqrt{1 + \left(\frac{6}{m}\right)^2(\omega_n - n)^2}}, \qquad \omega_n = \mathrm{sign}(n)\sqrt{\left(\frac{m}{6}\right)^2 + n^2}. \tag{93}
$$

In order to accomplish a realization of the $SU(2|2)$ algebra, we must identify a restricted set of level-I states upon which the algebra closes. These are states with one oscillator. We choose it to be a left-moving (untilded) oscillator, but the opposite choice is equally valid. The main point in uncovering the $SU(2|2)$ structure, as was discussed previously, is to relax the level-matching condition. We therefore do not consider any right-moving excitations. We define the (un-level-matched) states

$$
|\phi^{\dot\beta_2}\rangle_{\gamma_2} = \alpha^{\dot\beta_2}_{-n\,\gamma_2}|0\rangle, \qquad |\psi^{\dot\beta_1}\rangle_{\gamma_2} = \psi^{\dot\beta_1}_{-n\,\gamma_2}|0\rangle, \tag{94}
$$

[16] Since $Q^1_+$ and $Q^2_-$ are of different $SO(8)$ chirality, $Q^1_+ + iQ^2_-$ or $\gamma^4(Q^1_+ + iQ^2_-)$ mix $SU(2)$ representations and can not contribute to a $SU(2|2)$ supercharge.

where $\gamma_2$ is a spectator index which we subsequently drop and $n > 0$. We then find the standard $SU(2|2)$ action

$$\begin{aligned}
Q^{\dot\alpha_1}_{\dot\alpha_2}|\phi^{\dot\beta_2}\rangle &= a\,\delta^{\dot\beta_2}_{\dot\alpha_2}|\psi^{\dot\alpha_1}\rangle, \\
Q^{\dot\alpha_1}_{\dot\alpha_2}|\psi^{\dot\beta_1}\rangle &= b\,\epsilon^{\dot\alpha_1\dot\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}|\phi^{\dot\beta_2}\rangle, \\
\bar Q^{\dot\alpha_2}_{\dot\alpha_1}|\phi^{\dot\beta_2}\rangle &= c\,\epsilon^{\dot\alpha_2\dot\beta_2}\epsilon_{\dot\alpha_1\dot\beta_1}|\psi^{\dot\beta_1}\rangle, \\
\bar Q^{\dot\alpha_2}_{\dot\alpha_1}|\psi^{\dot\beta_1}\rangle &= d\,\delta^{\dot\beta_1}_{\dot\alpha_1}|\phi^{\dot\alpha_2}\rangle,
\end{aligned} \tag{95}$$

where we have that

$$\begin{aligned}
a &= 2i\frac{e^{i\pi/4}}{\sqrt{2\alpha' p^+}}\,\Omega_n\,\omega_n, \quad b = i\frac{e^{i\pi/4}}{\sqrt{2\alpha' p^+}}\,\Omega_{-n}, \\
c &= -2i\frac{e^{-i\pi/4}}{\sqrt{2\alpha' p^+}}\,\Omega_{-n}\,\omega_n, \quad d = -i\frac{e^{-i\pi/4}}{\sqrt{2\alpha' p^+}}\,\Omega_n.
\end{aligned} \tag{96}$$

We note that

$$\begin{aligned}
&\frac{1}{2}(ad+bc) = \frac{\omega_n}{\alpha' p^+}, \qquad ad - bc = \frac{\omega_n}{\alpha' p^+}\left(\Omega_n^2 - \Omega_{-n}^2\right) \simeq \frac{\beta}{3}, \\
&ab = (cd)^* = -\frac{i}{\alpha' p^+}\Omega_n\Omega_{-n}\omega_n \simeq -\frac{in}{\alpha' p^+},
\end{aligned} \tag{97}$$

where $\simeq$ indicates the $n \ll \alpha' p^+$ limit. We notice the consistency with our expectations: the energy of the state (i.e. $p^-$) is indeed given by $(ad+bc)/2$, while the central charge $ab$ is indeed proportional to the small $n$ limit of $(e^{-in/J}-1)$ with the consistent proportionality constant $R^2/\alpha'$ appearing in the energy[17]. Computing the $\{Q,\bar Q\}$ commutator using (92), we find (in the $n \ll \alpha' p^+$ limit)

$$\begin{aligned}
&\{Q^{\dot\alpha_1}_{\dot\alpha_2}, \bar Q^{\dot\beta_2}_{\dot\beta_1}\} = \delta^{\dot\alpha_1}_{\dot\beta_1}\delta^{\dot\beta_2}_{\dot\alpha_2}H + \frac{\beta}{3}\,\delta^{\dot\beta_2}_{\dot\alpha_2}\mathcal{L}^{\dot\alpha_1}_{\dot\beta_1} + \frac{\beta}{3}\,\delta^{\dot\alpha_1}_{\dot\beta_1}\mathcal{R}^{\dot\beta_2}_{\dot\alpha_2}, \\
&\{Q^{\dot\alpha_1}_{\dot\alpha_2}, Q^{\dot\beta_1}_{\dot\beta_2}\} = \epsilon^{\dot\alpha_1\dot\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}\mathcal{P}, \qquad \{\bar Q^{\dot\alpha_2}_{\dot\alpha_1}, \bar Q^{\dot\beta_2}_{\dot\beta_1}\} = \epsilon^{\dot\alpha_2\dot\beta_2}\epsilon_{\dot\alpha_1\dot\beta_1}\mathcal{K},
\end{aligned} \tag{98}$$

where[18]

$$\begin{aligned}
\mathcal{R}^{\dot\beta_2}_{\dot\alpha_2} &= \frac{1}{2}\sum_{n>0}\frac{1}{\omega_n}\left(\alpha_{-n}^{\dot\beta_2\gamma_2}\alpha_{n\,\dot\alpha_2\gamma_2} - \frac{1}{2}\delta^{\dot\beta_2}_{\dot\alpha_2}\,\alpha_{-n}^{\dot\gamma_2\gamma_2}\alpha_{n\,\dot\gamma_2\gamma_2} + \text{right-movers}\right), \\
\mathcal{L}^{\dot\alpha_1}_{\dot\beta_1} &= \sum_{n>0}\left(\psi^{\dot\alpha_1}_{-n\,\gamma_2}\psi^{\gamma_2}_{n\,\dot\beta_1} - \frac{1}{2}\delta^{\dot\beta_1}_{\dot\alpha_1}\psi^{\dot\gamma_1}_{-n\,\gamma_2}\psi^{\gamma_2}_{n\,\dot\gamma_1} + \text{right-movers}\right), \\
H &= \frac{1}{\alpha' p^+}\sum_{n>0}\left(\alpha^{i'}_{-n}\alpha^{i'}_n + \omega_n\,\psi^{\dot\gamma_1}_{-n\,\gamma_2}\psi^{\gamma_2}_{n\,\dot\gamma_1} + \text{right-movers}\right), \\
\mathcal{P} &= -\mathcal{K} = -\frac{i}{\alpha' p^+}\sum_{n>0} n\left(\frac{1}{\omega_n}\alpha^{i'}_{-n}\alpha^{i'}_n + \psi^{\dot\gamma_1}_{-n\,\gamma_2}\psi^{\gamma_2}_{n\,\dot\gamma_1} - \text{right-movers}\right).
\end{aligned} \tag{99}$$

[17] Here we have used $p^+ = J/R^2$, see (69).

[18] Supersymmetry ensures that the normal ordering constants in $H$ and $\mathcal{P}$ are zero.

Note that $\mathcal{P}$ is nothing but the level-matching operator (restricted to the $X^{i'}$ supermultiplet) as previously discussed. One can then verify that the action of $\mathcal{R}$ and $\mathcal{L}$ upon our states are

$$
\begin{aligned}
\mathcal{R}^{\dot\beta_2}_{\dot\alpha_2}|\phi^{\dot\gamma_2}\rangle &= \delta^{\dot\gamma_2}_{\dot\alpha_2}|\phi^{\dot\beta_2}\rangle - \frac{1}{2}\delta^{\dot\beta_2}_{\dot\alpha_2}|\phi^{\dot\gamma_2}\rangle, \\
\mathcal{L}^{\dot\alpha_1}_{\dot\beta_1}|\psi^{\dot\gamma_1}\rangle &= \delta^{\dot\gamma_1}_{\dot\beta_1}|\psi^{\dot\alpha_1}\rangle - \frac{1}{2}\delta^{\dot\alpha_1}_{\dot\beta_1}|\psi^{\dot\gamma_1}\rangle,
\end{aligned}
\tag{100}
$$

as they should be. Finally we note that the value of $ad - bc$ from (97) is what it needs to be in order to close the algebra. We have thus found the centrally extended $SU(2|2)$ algebra in the $n \ll \alpha' p^+$ limit of the string dual to SYM on $\mathbb{R} \times S^2$.

## 7.4 Generalizations, $S$-matrix, finite-size effects, and giant magnons

As discussed at the start of section 7, the IIA plane-wave appears in the BMN-like limit of the string duals of a rich class of vacua of any of the three theories: SYM on $\mathbb{R} \times S^2$, SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$, or the PWMM. Thus the $SU(2|2)$ algebra derived in the last section exists in all of these theories, as long as the vacuum for the model being studied is well separated. For analyses similar to those at the start of section 7, but for $\mathbb{R} \times S^3/\mathbb{Z}_k$ and the PWMM, see [7]. Having found the $SU(2|2)$ structure, it is natural to ask whether we can repeat the very rich battery of tests and analyses which have been carried out in the case of $AdS/CFT$ for $AdS_5 \times S^5$, assuming that our gauge theories really do posses an all-loop integrable sector. These include the matching of energies of spinning strings to the thermodynamic limit of the associated spin-chains, matching the worldsheet S-matrix to gauge theory, matching finite-size effects (i.e. $1/J$ corrections to the energies of states), and the existence of solitonic string configurations with very large worldsheet momentum, the giant magnons. In this section we will visit each of these issues in a qualitative manner, leaving any concrete analyses to further work.

### 7.4.1 Worldsheet $S$-matrix and finite-size effects

In order to discuss a worldsheet $S$-matrix, we must have an interacting sigma model. Since the plane-wave worldsheet theory is free, one must include curvature corrections in order to develop worldsheet interactions. The near plane-wave limit is complicated (as compared to the $AdS_5 \times S^5$ case [44]) by the dependence of the Lin-Maldacena geometries on the $\rho$ and $\eta$ coordinates (the coordinates $r$ and $\theta$ in (64)), i.e. the spatial coordinates transverse to the $S^2$ and $S^5$. The dilaton, B-field, and Ramond-Ramond field strengths develop dependence on these coordinates away from the strict plane-wave limit, i.e. their $\mathcal{O}(R^{-2})$ corrections are $\eta$ and $\rho$ dependent. However, despite these complications, if, as we expect, the $SU(2|2)$ symmetry is exact and so remains at $\mathcal{O}(R^{-2})$, then the $S$-matrix is highly constrained by this symmetry, to a single undetermined function $S^0_{12}$ (see (60)) [2]. If we consider the scattering of the bosonic $SO(4)$ excitations (the $X^{i'}$ of section 7.1) on a subset of states with no excitations from

the $SO(3)$ part of the geometry, then we will find the same relevant $S$-matrix elements as those found for the $AdS_5 \times S^5$ superstring in [45]. This is the trivial statement that both theories share an $S^5$ and so share its near plane-wave geometry. But then the function $S^0_{12}$ is determined at this order in the large-$R$ expansion. It would therefore not be surprising if the only change between $AdS_5 \times S^5$ and the theories considered here is that $\lambda$ is with replaced with $h(\lambda)$[19], i.e. that the same expression for the BES phase-factor [26] found in the expression for $S^0_{12}$ in $\mathcal{N} = 4$ SYM is the relevant one here with $\lambda \to h(\lambda)$. Further work would be required to verify (or disprove) this possibility. Similar statements apply to the finite-size corrections to the string spectrum. At leading order, these are given by first-order perturbation theory, and by the same logic, states built from bosonic $SO(4)$ excitations alone must share the same leading order finite-size corrections as those found in $AdS/CFT$. The non-trivial information comes at next-to-leading order, where second-order perturbation theory must be used, and where $SO(3)$ excitations appear in the intermediate states.

### 7.4.2 Spinning strings and giant magnons

Continuing our analogy with $AdS_5 \times S^5$ we may think about macroscopic spinning strings, corresponding to the thermodynamic limit of the gauge theory spin-chains. Again, the Lin-Maldacena geometries contain an $S^5$ which is the site of the $SU(2|2)$ symmetry. Any of the spinning string solutions with spins only in the $S^5$ may be borrowed from $AdS_5 \times S^5$. The only difference is the modified relationship between the radius of the $S^5$ and the gauge theory coupling, i.e. the strong coupling consequence of replacing $\lambda \to h(\lambda)$. The interesting question in this regard are the $1/R^2$ corrections to the energies of these spinning strings. A semi-classical calculation would require including the fluctuations of the $SO(3)$ modes, and these have a very different structure than the corresponding $AdS_5$ modes (c.f. [46]). It would be very interesting to attempt such a calculation, which would go towards fixing $h(\lambda)$ at next-to-leading order at strong coupling and give information on the form of the phase-factor in $S^0_{12}$. The giant magnon is of course also present in the Lin-Maldacena geometries, for the same trivial reason that there is an $S^5$ which will accommodate it. This is consistent with the strong-coupling limit of the $SU(2|2)$ dispersion relation (62). The finite-size correction to the energy of the giant magnon [47, 48] does not require a semi-classical treatment; the calculation takes place within the $\mathbb{R} \times S^2$ holding the magnon solution. Thus that correction is also valid for our case with $\lambda \to h(\lambda)$. This finite-size correction has also been obtained from the integrability of $\mathcal{N} = 4$ SYM through the Bethe ansatz [49]. There it is claimed that the match tests the form of the phase-factor appearing in $S^0_{12}$. Thus, we have another piece of evidence suggesting that the $\lambda \to h(\lambda)$ replacement may bring us from $\mathcal{N} = 4$ SYM, to the potentially integrable sector described here for SYM on $\mathbb{R} \times S^2$, SYM on $\mathbb{R} \times S^3/\mathbb{Z}_k$, and the PWMM.

[19]Of course there will be a different $h(\lambda)$ for each gauge theory and each vacua around which the expansion is being carried out.

# 8 Concluding remarks

In this paper we have found an application for the rich constraining power of the mass-deformed $SU(2|2)$ algebra in the gauge/gravity duality for $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$ and its dimensional reductions. We have mostly focused on the three dimensional $\mathcal{N}=8$ SYM on $\mathbb{R}\times S^2$ and its string dual to illustrate the use of this superalgebra in the computation of the spectrum of the gauge as well as the dual world-sheet theory. The two-loop gauge theory results and the leading order strong coupling computations done in the plane wave limit of the associated string theory suggest a potentially integrable $SU(2|3)$ sector for this particular realization of the gauge/gravity duality. Moreover, we find that various quantities, such as the form of the all-loop dispersion relation and the "matrix" structure of the S-matrix for the gauge theory Hamiltonian are exactly the same in this theory as the planar dilatation operator for $\mathcal{N}=4$ SYM.

Despite the methodological similarities, there are various fundamental differences that distinguish the three and four dimensional gauge theories from each other. For instance, analytical dependence of the physical spectrum on the effective 't Hooft couplings, encoded in the function $h(\lambda)$. Furthermore, the world sheet theory for the three dimensional gauge theory is not a coset model which makes the issue of understanding even its classical integrability challenging. Finally, the gauge theory appears to possess multiple vacuaa (which is reflected on the string theory side in the various disc configurations discussed earlier), which is unlike the physical behavior of the four dimensional superconformal theory. It is thus gratifying that despite these differences, various physical quantities can be analyzed in both these gauge theories using similar techniques of analysis. Apart from another venue for the potential utilization of the powerful algebraic methods tied to integrable structures, our study also opens up some interesting lines of investigation which we comment on below.

An obvious question to ask is whether or not the tell-tale signs of integrability for the three dimensional theory translate into all-loop integrability. Integrability to all orders, even in a restricted sub-sector of the theory (like the $SU(2)$ sector) would be a powerful boost towards performing a comprehensive test for the gauge/gravity duality without the use of conformal symmetries. Assuming integrability holds in a subsector of the gauge theory, the complete determination of the interpolating $h$ function, which we have computed at weak and strong coupling, is certainly an extremely interesting question and might be amenable to analysis by methods such as the Y-system, which has recently yielded dramatic results [50].

As mentioned before in the paper, a fuller understanding of the the gauge/gravity duality for the other dimensional reductions of the four dimensional gauge theory might be gained by adapting the present algebraic techniques accordingly. In particular, the interplay between the non-trivial vacua for the gauge theories in question and mass-deformed algebras is another potential line of investigation coming out of the present analysis.

Looking beyond the immediate concerns of this paper and the computation of spectra; the rôle of supersymmetry in the study of other extended degrees of freedom, such as Wilson loops would be another interesting problem to study. The massless version of the three dimensional theory defined on $\mathbb{R}^3$ has recently been shown to

posses a large class of BPS Wilson loops whose expectation values are completely determined by supersymmetry [51]. The investigation of the corresponding extended objects of the massive theory on $\mathbb{R} \times S^2$ should yield further valuable information for the field theory and its string theoretic counterpart.

On a final, somewhat tangential note, it is worth pointing out that massive algebras also arise in the context of three dimensional SYM theories even in flat backgrounds with minimal supersymmetry [52]. In these theories the supersymmetry algebra is deformed by the spacetime rotation group as opposed to the $R$-symmetry group. Furthermore, the mass-gap for the gluonic fields in these theories is generated by a term that is closely related to the volume measure on the gauge invariant configuration space for pure Yang-Mills theory in three dimensions [53] (for recent progress in three dimensional pure Yang-Mill theory on $\mathbb{R} \times S^2$, see [54]). This is perhaps indicative of a potential deeper connection between massive supersymmetry algebras and dynamical mass-generation in confining three dimensional SYM theories.

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# A Gamma matrices and mode expansions

The $SO(1,9)$ gamma matrices are given by [42]

$$\begin{aligned} &\Gamma^0 = -i\sigma^2 \otimes \mathbf{1}_{16}, \quad \Gamma^{11} = \sigma^1 \otimes \mathbf{1}_{16}, \quad \Gamma^I = \sigma^3 \otimes \gamma^I, \\ &\Gamma^9 = -\sigma^3 \otimes \gamma^9, \quad \Gamma^{\pm} = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^{11}), \end{aligned} \tag{101}$$

where $\gamma^9 = \gamma^{12345678}$, where $\gamma^I$ are the 16×16 $SO(8)$ gamma matrices. We choose the following representation for them

$$\gamma^I = \begin{pmatrix} 0 & \tilde{\gamma}^I_{a\dot{a}} \\ \tilde{\gamma}^I_{\dot{a}a} & 0 \end{pmatrix}, \tag{102}$$

where $a, \dot{a}$ run from 1 to 8. We decompose the $\tilde{\gamma}^I$ into two $SU(2) \times SU(2)$ representations (one for each $SO(4)$):

$$\tilde{\gamma}^i_{a\dot{a}} = \begin{pmatrix} 0 & \sigma^i{}_{\alpha_1\dot{\beta}_1}\delta^{\beta_2}_{\alpha_2} \\ \sigma^{i\,\dot{\alpha}_1\beta_1}\delta^{\dot{\alpha}_2}_{\dot{\beta}_2} & 0 \end{pmatrix}, \qquad \tilde{\gamma}^i_{\dot{a}a} = \begin{pmatrix} 0 & \sigma^i{}_{\alpha_1\dot{\beta}_1}\delta^{\dot{\alpha}_2}_{\dot{\beta}_2} \\ \sigma^{i\,\dot{\alpha}_1\beta_1}\delta^{\beta_2}_{\alpha_2} & 0 \end{pmatrix}, \tag{103}$$

$$\tilde{\gamma}^{i'}_{a\dot{a}} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'}{}_{\alpha_2\dot{\beta}_2} & 0 \\ 0 & \delta^{\dot{\alpha}_1}_{\dot{\beta}_1}\sigma^{i'\,\dot{\alpha}_2\beta_2} \end{pmatrix}, \qquad \tilde{\gamma}^{i'}_{\dot{a}a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'\,\dot{\alpha}_2\beta_2} & 0 \\ 0 & \delta^{\dot{\alpha}_1}_{\dot{\beta}_1}\sigma^{i'}{}_{\alpha_2\dot{\beta}_2} \end{pmatrix} \tag{104}$$

The $SU(2)$ indices $\alpha_i, \beta_i$ are raised and lowered using $\epsilon_{\alpha\beta}$, where $\epsilon_{12} = -\epsilon_{21} = 1$, and similarly for the dotted indices. The $\sigma$-matrices satisfy the relations

$$\sigma^i_{\alpha\dot\alpha}\sigma^{j\,\dot\alpha\beta} + \sigma^j_{\alpha\dot\alpha}\sigma^{i\,\dot\alpha\beta} = 2\delta^{ij}\delta^\beta_\alpha\,, \qquad \sigma^{i\,\dot\alpha\alpha}\sigma^j_{\alpha\dot\beta} + \sigma^{j\,\dot\alpha\alpha}\sigma^i_{\alpha\dot\beta} = 2\delta^{ij}\delta^{\dot\alpha}_{\dot\beta}\,. \tag{105}$$

Some other properties satisfied by these matrices are

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = \delta^\delta_\alpha\delta^\gamma_\beta - \delta^\gamma_\alpha\delta^\delta_\beta\,, \tag{106}$$

$$\sigma^i_{\alpha\dot\beta}\sigma^{j\,\dot\beta}{}_\beta = -\delta^{ij}\epsilon_{\alpha\beta} + \sigma^{ij}_{\alpha\beta}\,, \qquad (\sigma^{ij}_{\alpha\beta} \equiv \sigma^{[i}_{\alpha\dot\alpha}\sigma^{j]\,\dot\alpha}{}_\beta = \sigma^{ij}_{\beta\alpha}) \tag{107}$$

$$\sigma^i_{\alpha\dot\alpha}\sigma^{j\,\alpha}{}_{\dot\beta} = -\delta^{ij}\epsilon_{\dot\alpha\dot\beta} + \sigma^{ij}_{\dot\alpha\dot\beta}\,, \qquad (\sigma^{ij}_{\dot\alpha\dot\beta} \equiv \sigma^{[i}_{\alpha\dot\alpha}\sigma^{j]\,\alpha}{}_{\dot\beta} = \sigma^{ij}_{\dot\beta\dot\alpha}) \tag{108}$$

$$\sigma^k_{\alpha\dot\alpha}\sigma^k_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,, \tag{109}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\gamma\delta} = 4(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma})\,, \tag{110}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\dot\gamma\dot\delta} = 0\,, \tag{111}$$

$$2\sigma^i_{\alpha\dot\alpha}\sigma^j_{\beta\dot\beta} = \delta^{ij}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \sigma^{k(i}_{\alpha_1\beta_1}\sigma^{j)k}_{\dot\alpha_1\dot\beta_1} - \epsilon_{\alpha\beta}\sigma^{ij}_{\dot\alpha\dot\beta} - \sigma^{ij}_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,. \tag{112}$$

The following Fierz identities are also useful for calculating the commutator of the supercharges,

$$\left(\psi^A_\pm\right)_\gamma \left(\psi^A_\pm\right)_\delta = \frac{1}{4}\delta_{\gamma\delta}\psi^A_\pm\psi^A_\pm + \frac{1}{8}\gamma^{ij}_{\gamma\delta}\,\psi^A_\pm\gamma^{ij}\psi^A_\pm + \frac{1}{8}\gamma^{i'j'}_{\gamma\delta}\,\psi^A_\pm\gamma^{i'j'}\psi^A_\pm, \tag{113}$$

$$\left(\psi^A_\pm\right)_\gamma \left(\psi^B_\mp\right)_\delta = \frac{1}{4}\gamma^i_{\gamma\delta}\,\psi^A_\pm\gamma^i\psi^B_\mp + \frac{1}{4}\gamma^{i'}_{\gamma\delta}\,\psi^A_\pm\gamma^{i'}\psi^B_\mp, \tag{114}$$

where no summation is implied on the $A, B$ indices and $\gamma, \delta$ are $SO(8)$ spinor indices running from 1 to 16.

The mode expansions (excluding zero-modes), for the string embedding functions are given by [41]

$$\begin{aligned}
&X^{i'} = i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{\omega_n}\left(\alpha^{i'}_n\phi^*_n + \tilde\alpha^{i'}_n\phi_n\right), \quad \dot X^{i'} = \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\alpha^{i'}_n\phi^*_n + \tilde\alpha^{i'}_n\phi_n\right),\\
&\psi^{2\ \dot\alpha_1}_{-\alpha_2} = \sum_{n\neq 0} c_n\left(\psi_{n\,\alpha_2}^{\ \dot\alpha_1}\,\phi^*_n - i\frac{6}{m}(\omega_n - n)\sigma^{4\,\dot\alpha_1\gamma_1}\tilde\psi_{n\,\gamma_1\alpha_2}\,\phi_n\right),\\
&\psi^1_{+\alpha_1\alpha_2} = \sum_{n\neq 0} c_n\left(\psi_{n\,\alpha_1\alpha_2}\,\phi^*_n + i\frac{6}{m}(\omega_n - n)\sigma^4{}_{\alpha_1\dot\gamma_1}\tilde\psi_{n\ \alpha_2}^{\ \dot\gamma_1}\,\phi_n\right),
\end{aligned} \tag{115}$$

where

$$\begin{aligned}
&\omega_n = \mathrm{sign}(n)\sqrt{\left(\frac{m}{6}\right)^2 + n^2}, \quad \phi_n = e^{in\sigma}, \quad c_n = \frac{\sqrt{\alpha'}}{\sqrt{1 + \left(\frac{6}{m}\right)^2(\omega_n - n)^2}},\\
&\{\psi_{n\,\alpha_2}^{\ \dot\alpha_1}, \psi_{m\ \dot\beta_1}^{\ \beta_2}\} = \delta_{n+m}\delta^{\dot\alpha_1}_{\dot\beta_1}\delta^{\beta_2}_{\alpha_2}, \quad \{\tilde\psi_{n\ \alpha_2}^{\ \alpha_1}, \tilde\psi_{m\ \beta_1}^{\ \beta_2}\} = \delta_{n+m}\delta^{\alpha_1}_{\beta_1}\delta^{\beta_2}_{\alpha_2},\\
&[\alpha^{i'}_n, \alpha^{j'}_m] = \omega_n\delta_{n+m}\delta^{i'j'}, \quad [\tilde\alpha^{i'}_n, \tilde\alpha^{j'}_m] = \omega_n\delta_{n+m}\delta^{i'j'},
\end{aligned} \tag{116}$$

and so negative mode numbers represent creation operators.

## B Relation between $SO(6)$ and $SU(4)$

The $SO(9,1)$ gamma matrices are given by

$$\Gamma^\mu = \gamma^\mu \otimes 1_8, \quad \Gamma^{AB} = \gamma_5 \otimes \begin{pmatrix} 0 & -\tilde{\rho}^{AB} \\ \rho^{AB} & 0 \end{pmatrix} = -\Gamma^{BA}. \tag{117}$$

$\Gamma^{AB}$ satisfies $\{\Gamma^{AB}, \Gamma^{CD}\} = \epsilon^{ABCD}$. $\rho^{AB}$ and $\tilde{\rho}^{AB}$ are defined by

$$(\rho^{AB})_{CD} = \delta^A_C \delta^B_D - \delta^A_D \delta^B_C, \quad (\tilde{\rho}^{AB})^{CD} = \frac{1}{2}\epsilon^{CDEF}(\rho^{AB})_{EF} = \epsilon^{ABCD}. \tag{118}$$

The relationship between the $SO(6)$ and $SU(4)$ bases is

$$\begin{aligned} &X^{AB} = \frac{1}{2}\epsilon^{ABCD}X_{CD}, \quad X^{AB} = -X^{BA} = X^\dagger_{AB}, \quad X_{i4} = \frac{1}{2}(X_i + iX_{i+3}) \\ &\Gamma^{i4} = \frac{1}{2}(\Gamma^i - i\Gamma^{i+3}), \quad X_{AB}\Gamma^{AB} = X_m\Gamma^m. \end{aligned} \tag{119}$$

## References


[1] N. Beisert, "The su(2—3) dynamic spin chain," *Nucl. Phys.* **B682** (2004) 487–520, `arXiv:hep-th/0310252`.

[2] N. Beisert, "The su(2—2) dynamic S-matrix," *Adv. Theor. Math. Phys.* **12** (2008) 945, `arXiv:hep-th/0511082`.

[3] N. Beisert, "The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2—2) Symmetry," *J. Stat. Mech.* **0701** (2007) P017, `arXiv:nlin/0610017`.

[4] D. Gaiotto, S. Giombi, and X. Yin, "Spin Chains in N=6 Superconformal Chern-Simons-Matter Theory," *JHEP* **04** (2009) 066, `arXiv:0806.4589 [hep-th]`.

[5] J. A. Minahan and K. Zarembo, "The Bethe ansatz for superconformal Chern-Simons," *JHEP* **09** (2008) 040, `arXiv:0806.3951 [hep-th]`.

[6] A. Agarwal, N. Beisert, and T. McLoughlin, "Scattering in Mass-Deformed N¿=4 Chern-Simons Models," *JHEP* **06** (2009) 045, `arXiv:0812.3367 [hep-th]`.

[7] H. Lin and J. M. Maldacena, "Fivebranes from gauge theory," *Phys. Rev.* **D74** (2006) 084014, `arXiv:hep-th/0509235`.

[8] N. Kim, T. Klose, and J. Plefka, "Plane-wave matrix theory from N = 4 super Yang-Mills on R x S**3," *Nucl. Phys.* **B671** (2003) 359–382, `arXiv:hep-th/0306054`.

[9] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, "An index for 4 dimensional super conformal theories," *Commun. Math. Phys.* **275** (2007) 209–254, arXiv:hep-th/0510251.

[10] S. Fubini, A. J. Hanson, and R. Jackiw, "New approach to field theory," *Phys. Rev.* **D7** (1973) 1732–1760.

[11] G. Arutyunov, S. Frolov, J. Plefka, and M. Zamaklar, "The off-shell symmetry algebra of the light-cone AdS(5) x S**5 superstring," *J. Phys.* **A40** (2007) 3583–3606, arXiv:hep-th/0609157.

[12] G. Ishiki, S. Shimasaki, Y. Takayama, and A. Tsuchiya, "Embedding of theories with SU(2—4) symmetry into the plane wave matrix model," *JHEP* **11** (2006) 089, arXiv:hep-th/0610038.

[13] G. Grignani, L. Griguolo, N. Mori, and D. Seminara, "Thermodynamics of theories with sixteen supercharges in non-trivial vacua," *JHEP* **10** (2007) 068, arXiv:0707.0052 [hep-th].

[14] C. W. H. Lee and S. G. Rajeev, "Symmetries of large N(c) matrix models for closed strings," *Phys. Rev. Lett.* **80** (1998) 2285–2288, arXiv:hep-th/9711052.

[15] C. W. H. Lee and S. G. Rajeev, "A Lie algebra for closed strings, spin chains and gauge theories," *J. Math. Phys.* **39** (1998) 5199–5230, arXiv:hep-th/9806002.

[16] C. W. H. Lee and S. G. Rajeev, "A review of symmetry algebras of quantum matrix models in the large-N limit," *Int. J. Mod. Phys.* **A14** (1999) 4395–4456, arXiv:hep-th/9906060.

[17] A. Agarwal and S. G. Rajeev, "Yangian symmetries of matrix models and spin chains: The dilatation operator of N = 4 SYM," *Int. J. Mod. Phys.* **A20** (2005) 5453–5490, arXiv:hep-th/0409180.

[18] T. Klose and J. Plefka, "On the integrability of large N plane-wave matrix theory," *Nucl. Phys.* **B679** (2004) 127–142, arXiv:hep-th/0310232.

[19] T. Fischbacher, T. Klose, and J. Plefka, "Planar plane-wave matrix theory at the four loop order: Integrability without BMN scaling," *JHEP* **02** (2005) 039, arXiv:hep-th/0412331.

[20] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," *JHEP* **03** (2003) 013, arXiv:hep-th/0212208.

[21] G. Ishiki, Y. Takayama, and A. Tsuchiya, "N = 4 SYM on R x S**3 and theories with 16 supercharges," *JHEP* **10** (2006) 007, arXiv:hep-th/0605163.

[22] A. Agarwal and G. Ferretti, "Higher charges in dynamical spin chains for SYM theory," *JHEP* **10** (2005) 051, arXiv:hep-th/0508138.

[23] N. Beisert, C. Kristjansen, and M. Staudacher, “The dilatation operator of N = 4 super Yang-Mills theory,” *Nucl. Phys.* **B664** (2003) 131–184, `arXiv:hep-th/0303060`.

[24] N. Beisert, T. McLoughlin, and R. Roiban, “The Four-Loop Dressing Phase of N=4 SYM,” *Phys. Rev.* **D76** (2007) 046002, `arXiv:0705.0321 [hep-th]`.

[25] F. Fiamberti, A. Santambrogio, and C. Sieg, “Five-loop anomalous dimension at critical wrapping order in N=4 SYM,” `arXiv:0908.0234 [hep-th]`.

[26] N. Beisert, B. Eden, and M. Staudacher, “Transcendentality and crossing,” *J. Stat. Mech.* **0701** (2007) P021, `arXiv:hep-th/0610251`.

[27] N. Beisert, “The S-Matrix of AdS/CFT and Yangian Symmetry,” *PoS* **SOLVAY** (2006) 002, `arXiv:0704.0400 [nlin.SI]`.

[28] I. V. Aniceto, “Matrix Reduction and the su(2—2) superalgebra in AdS/CFT,” *Phys. Rev.* **D79** (2009) 086014, `arXiv:0809.2645 [hep-th]`.

[29] J. M. Maldacena, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, “Transverse fivebranes in matrix theory,” *JHEP* **01** (2003) 038, `arXiv:hep-th/0211139`.

[30] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” *Phys. Rev.* **D58** (1998) 046004, `arXiv:hep-th/9802042`.

[31] J. Bagger and N. Lambert, “Modeling multiple M2’s,” *Phys. Rev.* **D75** (2007) 045020, `arXiv:hep-th/0611108`.

[32] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” *Phys. Rev.* **D77** (2008) 065008, `arXiv:0711.0955 [hep-th]`.

[33] J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” *JHEP* **02** (2008) 105, `arXiv:0712.3738 [hep-th]`.

[34] A. Gustavsson, “Algebraic structures on parallel M2-branes,” *Nucl. Phys.* **B811** (2009) 66–76, `arXiv:0709.1260 [hep-th]`.

[35] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” *JHEP* **10** (2008) 091, `arXiv:0806.1218 [hep-th]`.

[36] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, “N=4 Superconformal Chern-Simons Theories with Hyper and Twisted Hyper Multiplets,” *JHEP* **07** (2008) 091, `arXiv:0805.3662 [hep-th]`.

[37] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, “N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds,” *JHEP* **09** (2008) 002, `arXiv:0806.4977 [hep-th]`.

[38] J. Gomis, A. J. Salim, and F. Passerini, "Matrix Theory of Type IIB Plane Wave from Membranes," *JHEP* **08** (2008) 002, arXiv:0804.2186 [hep-th].

[39] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, and H. Verlinde, "A Massive Study of M2-brane Proposals," *JHEP* **09** (2008) 113, arXiv:0807.1074 [hep-th].

[40] K. Sugiyama and K. Yoshida, "Type IIA string and matrix string on pp-wave," *Nucl. Phys.* **B644** (2002) 128–150, arXiv:hep-th/0208029.

[41] S.-j. Hyun and H.-j. Shin, "Solvable N = (4,4) type IIa string theory in plane-wave background and D-branes," *Nucl. Phys.* **B654** (2003) 114–134, arXiv:hep-th/0210158.

[42] S.-j. Hyun and H.-j. Shin, "N = (4,4) type IIA string theory on pp-wave background," *JHEP* **10** (2002) 070, arXiv:hep-th/0208074.

[43] A. Pankiewicz, "An alternative formulation of light-cone string field theory on the plane wave," *JHEP* **06** (2003) 047, arXiv:hep-th/0304232.

[44] C. G. Callan, Jr. *et al.*, "Quantizing string theory in AdS(5) x S**5: Beyond the pp- wave," *Nucl. Phys.* **B673** (2003) 3–40, arXiv:hep-th/0307032.

[45] T. Klose, T. McLoughlin, R. Roiban, and K. Zarembo, "Worldsheet scattering in AdS(5) x S**5," *JHEP* **03** (2007) 094, arXiv:hep-th/0611169.

[46] A. A. Tseytlin, "Spinning strings and AdS/CFT duality," arXiv:hep-th/0311139.

[47] G. Arutyunov, S. Frolov, and M. Zamaklar, "Finite-size effects from giant magnons," *Nucl. Phys.* **B778** (2007) 1–35, arXiv:hep-th/0606126.

[48] D. Astolfi, V. Forini, G. Grignani, and G. W. Semenoff, "Gauge invariant finite size spectrum of the giant magnon," *Phys. Lett.* **B651** (2007) 329–335, arXiv:hep-th/0702043.

[49] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling – the giant magnon," *Phys. Rev.* **D76** (2007) 126008, arXiv:0708.2208 [hep-th].

[50] N. Gromov, V. Kazakov, and P. Vieira, "Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory," *Phys. Rev. Lett.* **103** (2009) 131601, arXiv:0901.3753 [hep-th].

[51] A. Agarwal and D. Young, "Supersymmetric Wilson Loops in Diverse Dimensions," *JHEP* **06** (2009) 063, arXiv:0904.0455 [hep-th].

[52] A. Agarwal, "Mass Deformation of Super Yang-Mills in D=2+1," *Phys. Rev.* **D80** (2009) 105020, arXiv:0909.0959 [hep-th].

[53] D. Karabali, C.-j. Kim, and V. P. Nair, "Planar Yang-Mills theory: Hamiltonian, regulators and mass gap," *Nucl. Phys.* **B524** (1998) 661–694, arXiv:hep-th/9705087.

[54] A. Agarwal and V. P. Nair, "The Hamiltonian Analysis for Yang-Mills Theory on $R \times S^2$," *Nucl. Phys.* **B816** (2009) 117–138, arXiv:0807.2131 [hep-th].